Nonlinear Control Systems

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Definition

Nonlinear control systems are those control systems where nonlinearity plays a significant role, either in the controlled process (plant) or in the controller itself. Nonlinear plants arise naturally in numerous engineering and natural systems, including mechanical and biological systems, aerospace and automotive control, industrial process control, and many others.

Nonlinear control theory is concerned with the analysis and design of nonlinear control systems. It is closely related to nonlinear systems theory in general, which provides its basic analysis tools.

Characteristics

Numerous methods and approaches exist for the analysis and design of nonlinear control systems. A brief and informal description of some prominent ones is given next. Full details may be found in the textbooks [1-6], and in the Control Handbook [7].

Most of the theory and practice focuses on feedback control. A typical layout of a feedback control system is shown in Figure 1.
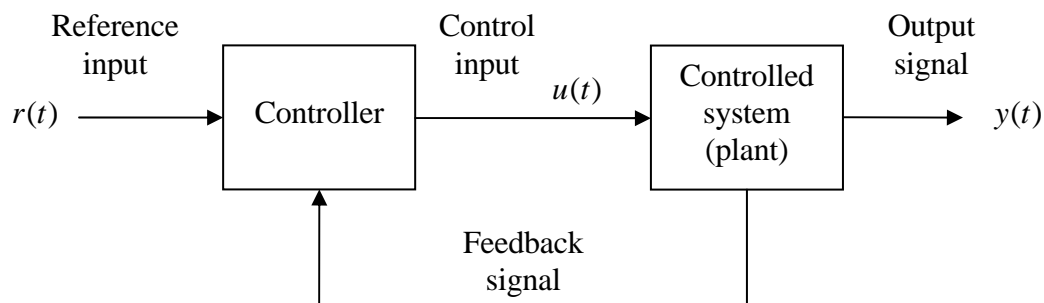

Figure 1 Basic feedback control system.

A basic (finite dimensional, time invariant) nonlinear system with continuous time parameter may be specified by the state-space model:

$$\tfrac{d}{dt} x(t) = f(x(t), u(t)); \quad x(0) = x_0$$

$$y(t) = h(x(t), u(t))$$

or, more succinctly, $\dot{x} = f(x, u)$ (the state equations) and $y = h(x, u)$ (the output equation).

Here $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the vector of input signals, and $y(t) \in \mathbb{R}^q$ is the output vector. This model may pertain to the plant (see Figure 1), as well as to the controller (with appropriately modified inputs and outputs). The state of the overall feedback system is then the combined state of the plant and the controller. A specific class of systems that has been studies in depth is linear-in-control systems, where $f(x, u) = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i$. We limit the discussion here to continuous-time systems, although similar theory exists for the discrete-time case.

Nonlinear models may be classified into smooth and non-smooth ones. The latter are often associated with parasitic effects such as dry friction and actuator saturation. When significant, these effects may enter as constraints in the design, or even require specific compensation techniques. Our discussion below pertains mainly to smooth nonlinearities.

Basic Concepts from Systems Theory

The following notions from systems theory are of particular importance and relevance to nonlinear control, and are dealt with in depth in the cited texts.

a. Equilibrium points: For the nonlinear system $\dot{x} = f(x)$, a point x^e in the state space is an equilibrium point if $f(x_e) = 0$. Similarly, for the controlled system $\dot{x} = f(x, u)$, the pair (x_e, u_e) is an equilibrium point if $f(x_e, u_e) = 0$.

b. Lyapunov stability: This is the basic notion of stability that deals with the asymptotic behavior of trajectories that start off an equilibrium point. An equilibrium point x_e of the system $\dot{x} = f(x)$ is (weakly) *stable* if all solutions $x(t)$ that start near x^e stay near it forever. It is *asymptotically stable* if, in addition, $x(t)$ converges to x_e ($\lim_{t \to \infty} x(t) = x_e$) whenever started near enough to it. If this convergence occurs for any initial state then x_e is *globally* asymptotically stable. Exponential stability requires an exponential rate of convergence to the equilibrium. Note that a nonlinear system may have several equilibrium points, each with different stability properties. For input-driven state equations with unspecified input, namely $\dot{x} = f(x, u)$, these stability notions are generalized by the concept of *input-to-state stability*, which requires the state vector to be close to equilibrium whenever both the initial state and the control input $u(t)$ are close to their equilibrium values.

c. Lyapunov's direct method: The most general approach to date for stability analysis of nonlinear systems is Lyapunov's method, which relies on the concept of a Lyapunov function or generalized energy function. Essentially, a Lyapunov function for an

equilibrium point x_e of the system $\dot{x} = f(x)$ is a differentiable function $V(x)$ which has a strict minimum as x_e, and so that its derivative $\dot{V}(x) \triangleq \frac{\partial V(x)}{\partial x} \cdot f(x)$ along the system trajectories is negative in some neighborhood of the equilibrium. Many extensions and refinement of this result exist, covering various stability properties such as exponential stability, global stability, and estimates on the domain of attraction of the equilibrium.

We note that various converse theorems establish the existence of a Lyapunov function whenever the equilibrium point is stable (in the appropriate sense); however no general procedure exists for finding such a function.

d. Linearization: The small-signal behavior of the system (1) around an equilibrium point (x_e, u_e) may be captured through a linear state equation of the form: $\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$, where $\tilde{x}(t) = x(t) - x_e$, $\tilde{u}(t) = u(t) - u_e$, and the matrices (A, B) are computed as corresponding gradients of the system function $f(x, u)$ at (x_e, u_e). A similar relation holds for the output equation. A basic stability result (also known as Lyapunov's indirect method) is that the Hurwitz-stability of the matrix A (namely, all eigenvalues with strictly negative real part) implies the asymptotic stability of the respective equilibrium point.

e. Input-output stability and gain: The dynamic system (1) is said to be input-output stable with respect to a signal norm $\|\cdot\|$, if $\| y(\cdot) \| \leq \gamma \| u(\cdot) \| + \beta$ for some constants $\gamma \geq 0$ and β (and every input $u(\cdot)$ in the input space). The name BIBO stable is also used when the norm is the max norm, namely $\| y(\cdot) \| = \sup_t \| y(t) \|$. The system gain is the smallest number γ that satisfied the above bound. For state-space models, various results relate input-output stability to corresponding stability properties of the state.

f. Passivity: The system-theoretic notion of passivity essentially captures the physical notion of a system that does not generate energy. As such, many mechanical and other systems satisfy this property. Passivity provides a useful analysis tool, and a basis for design methods. Notably, passivity implies stability, and the feedback connection of passive systems is passive.

g. Controllability and Reachability: These two closely-related concepts that apply to the state equation $\dot{x} = f(x, u)$ concern the possibility of reaching a given state from any other state (controllability) or reaching any other state from a given state (reachability) by choosing appropriate controls. Local versions focus on small neighborhoods of any given point. These properties have been studied in depth, especially for the class of linear-in-control systems, using tools from differential geometry.

h. Observability: Observability concerns the ability to distinguish between two (initial) states based on proper choice of input and observation of the system output. This concept, roughly, indicates whether a feedback controller that uses only the output y can fully control the state dynamics.

Analysis of Feedback Systems

Alongside general tools and methods from system theory, a number of results and analysis methods apply specifically to feedback systems, and some of these are described next. The basic feedback connection of two subsystems is shown in Figure 2.
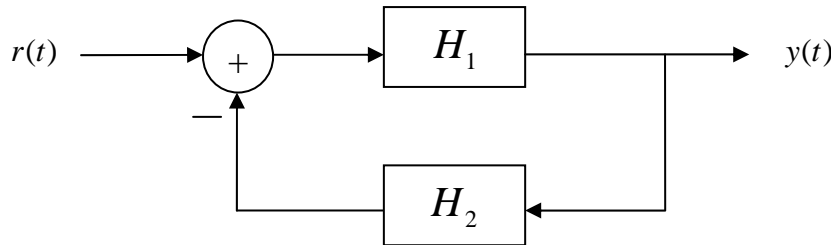

Figure 2: Negative feedback connection

a. Limit Cycles and Describing Function Analysis: Limit cycles, or sustained oscillations, are common in nonlinear feedback systems, and are usually not desired in control systems. The describing function method checks for the possibility of oscillations by (1) approximating the response of nonlinear elements to sinusoidal inputs of given amplitude and frequency by their first harmonics only (2) checking for the possibility of a loop gain of 1 with phase shift of 180 degrees (the so called harmonic balance equation, for a negative feedback system). In case of a positive answer the analysis yields estimates for the frequency and amplitude of oscillations. While the method is essentially heuristic it is often useful for initial analysis.

b. Small Gain Theorem: The small gain theorem allows to establish the input-output stability of a feedback system from properties its subsystems. Assume that H_1 and H_2 are both input-output stable, with respective upper-bounds γ_1 and γ_2 on their gains. If $\gamma_1\gamma_2 < 1$, then the feedback system in Figure 2 is input-output stable, and its gain is bounded by $\gamma_1/(1-\gamma_1\gamma_2)$.

c. Circle Criterion: Consider the special case H_1 is a linear time-invariant system, and H_2 is a static nonlinearity $h_2(\cdot)$ that satisfies a $[k_1,k_2]$ sector condition; in the scalar case this means that $h_2(x)/x \in [k_1,k_2]$. The circle criterion (and the related Popov criterion) provides frequency-domain conditions on the transfer function of H_1 that imply the stability of the feedback system.

Design methods

Control system design in general aims to satisfy certain performance objectives, such as stability, accurate input tracking, disturbance rejection, and robustness or insensitivity to parameter uncertainty (see the Control section for further details). The diverse nature of nonlinear systems necessarily calls for a variety of design approaches of different nature, and some of the more notable ones are briefly described below.

One design viewpoint is to view the controlled system as an approximately linear one, or linearize the system by appropriate transformation, to which well-established linear control techniques may be applied.

a. PID Control: The PID (Proportional-Integral-Derivative) regulator is a simple linear controller, which is often cited as the most prevalent feedback controller. In particular, it finds use in many non-linear applications, from industrial process control to robotic manipulators. On-site tuning of the PID controller parameters is often used, especially in the process control industry, and numerous manual and auto-tuning procedures exist based on direct measurement of some characteristics of the system response. Analytical (model-based) design is of course also used, often building on one of the linearization methods below.

b. Local linearization and Gain Scheduling: The simplest analytical approach to controller design for a nonlinear system relies on fitting an approximate linear model to the controlled system, usually through local linearization around a typical working point (see above), and then designing a linear controller for this model. Evidently this method may fail when non-linear effects are significant. Gain Scheduling takes this approach a step further: Linear controllers are designed for a range of possible operating points and conditions, and the appropriate controller is put into play according to the current system state.

c. Feedback Linearization: Feedback (or global) linearization uses input and state variable transformations to arrive to an equivalent linear system. As a simple example, the scalar system $\dot{x} = u^3 + f(x)$ is readily transformed to $\dot{x} = v$ by defining an auxiliary input $v = u^3 + f(x)$. A control law to determine v can now be designed for the linear system, and the actual control u may then be computed using the inverse relation $u = (v - f(y))^{\frac{1}{3}}$. The latter equation, which is in the form of state feedback, given the method its name. One can distinguish between full state linearization, where the state equation is fully linearized, and input-output linearization, where the input-output map is linearized. In either case, measurement of the entire state vector is required to implement the transformation. The theory provides conditions under which feedback linearization is possible, and procedures to compute the required transformations.

The following methods approach the design problem directly using non-linear tools, notably Lyapunov stability and Lyapunov functions, and are notable examples of *robust* nonlinear control.

d. Lyapunov Design and Redesign: In Lyapunov-based design, a stable system is synthesized by first choosing a candidate Lyapunov function V, and then selecting a state-feedback control law that renders the derivative of V negative. The Lyapunov redesign method provides the system with robustness to (bounded) uncertainly in the system dynamics. It starts with a stabilizing control law and Lyapunov function for the nominal system, and adds certain (non-smooth) terms to the control that ensure stability in the face of all admissible uncertainties. While Lyapunov redesign is restricted systems that satisfy a matching condition, so that the uncertainty terms enter the state equations at the same point as the control input, the basic approach has been extended to more general situations using recursive or *backstepping* methods.

e. Sliding Mode Control: In this robust design approach, also called Variable Structure Control, an appropriate manifold (often a linear surface) in the state space is first located on which the system dynamics takes a simple and stable form. This manifold is called the sliding surface or the switching surface. The control law is designed to force trajectories to reach that manifold in finite time, and stay there thereafter. As the basic control law is discontinuous by design around the switching surface, unwanted chattering around that may result and often require some smoothing of the control law.

Many other techniques from control engineering are applicable to the design of nonlinear systems, some of which may be considered as separate fields of control engineering. Among these we mention:

- Optimal Control: Here the control objective is to minimize a pre-determined cost function. The basic solution tools are Dynamic Programming and variational methods (Calculus of Variations and Pontryagin's maximum principle). The available solutions for nonlinear problems are mostly numeric.

- Model Predictive Control: An approximation approach to optimal control, where the control objective is optimized on-line for a finite time horizon. Due to computational feasibility this method has recently found wide applicability, mainly in industrial process control.

- Adaptive Control: A general approach to handle uncertainty and possible time variation of the controlled system model. Here the controller parameters are tuned on-line as part of the controller operation, using various estimation and learning techniques.

- Neural Network Control: A particular class of adaptive control systems, where the controller is in the form of an Artificial Neural Network.

- Fuzzy Logic Control: Here the controller implements an (often heuristic) set of logical (or discrete) rules for synthesizing the control signal based on the observed outputs. Defuzzification and fuzzification procedures are used to obtain a smooth control law from discrete rules.

A detailed description of these are related approaches, which are often considered as separate fields of control engineering, may be found in [7].

References

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Nonlinear Control Systems
Glossary of terms

Nonlinear control systems: The general class of (usually closed-loop) control systems where nonlinearity plays a significant role, either in the controlled process or in the controller itself.

System – Nonlinear: The general class of systems for which the relation between the input variables (initial conditions or external inputs) and output or state variables in not linear. Common descriptions are by (nonlinear) state equations, differential equations, or difference equations.

Equilibrium Points: In system theory, those points in the state space in which the time derivative of the state is null.

Lyapunov Stability: In system theory, the basic notion of stability in the state space, that deals with the asymptotic convergence to equilibrium of trajectories that start off an equilibrium point.

Lyapunov Function: An energy-like function that is used to establish stability properties of state-space systems, via Lyapunov's direct method.

Describing Function Analysis: An approximate analysis method used to determine the possibility and parameters of persistent oscillations in a closed-loop (feedback) system.

PID Control: A popular closed-loop control method that uses a simple (Proportional-Integral-Derivative) linear controller with easily tunable parameters.

Gain Scheduling: An approach to the control of non-linear systems, where different (often linear) controllers are used depending on some measure of the current system state.

Linearization: An approximate analysis approach that studies the dynamics of nonlinear systems through their linear approximation.

Global Linearization: An approach to the design of nonlinear control systems, where the controlled system is first transformed into an equivalent linear system using variable transformations. Also referred to as feedback linearization.

Feedback Linearization: See Global Linearization.

Lyapunov Design: A general approach to the design of nonlinear control systems, which starts with a candidate Lyapunov function and chooses feedback control to obtain desired properties for this function that guarantee stability and related properties.

Sliding Mode Control: An approach to the synthesis of feedback controllers for nonlinear control systems, where the system trajectories are forced to reach is finite time a certain desirable surface in the state space.