Decentralized Rate Regulation in Random Access Channels

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***Abstract*—We consider a time-slotted multipacket reception channel, shared by a finite number of mobile users who transmit to a common base station. Each user is allocated a fixed data rate, which may be imposed by the base station or self-determined. For sustaining the required rate over time, each user may adjust a single parameter which determines the individual transmission probability in a given slot. An equilibrium point is attained when the assigned data rates are met with equality. This paper analyzes the equilibrium points which result in this system, with a focus on power efficiency of the solution.**

While multiple equilibrium points exist in general, we establish that one of these equilibria is best for all users, in the sense that the transmission probability (hence the power investment) of each user is minimal. Further to the existence of worse equilibrium points, we point to the possibility of a partial-equilibrium with starvation, where stronger users (in terms of received power) satisfy their data rates, while preventing weaker ones from obtaining their respective rates. To avoid these sub-optimal working points, we suggest a distributed mechanism that converges to the best equilibrium point. Further analysis is provided for a specific channel model which involves perfect capture.

I. Introduction

A. Background and Motivation

The emerging use of wireless technologies (such as WIFI and WIMAX) for data communication has brought to focus the resource allocation and management task, with the objective of satisfying mobile users with heterogenous requirements. An effective management scheme needs to cope with scarce wireless resources such as bandwidth and power. On top of that, wireless channel conditions are often time-varying (e.g., due to channel fading [1]), and are greatly affected by network topology and mobility of individual nodes. As a result, efficient resource management becomes a complicated task.

Centralized scheduling algorithms have been suggested (e.g., [2]–[4] and references therein) to allow for QoS differentiation in non-stationary wireless platforms, and also support diverse fairness criteria such as maxmin and proportional fairness. These algorithms are designed to accommodate the additional wireless specific features. To understand the complexity involved, consider a single–cell uplink model where all mobile stations should be given an equal rate share. While scheduling the mobiles' transmission, a scheduler must consider the mobile's location (specifically, the distance from the base station), and also its power capabilities and current channel quality. Hence, a high-quality scheduler requires substantial centralized data and computational resources, and might become hard to implement in some wireless domains. Decentralized MAC algorithms represent a significantly different viewpoint for resource management. The classical Aloha protocol was designed at the early 70's as a unified distributed mechanism which can allow efficient media sharing. This protocol and its variants, such as CSMA-CD and tree-algorithms [5], have gained prominence due to their relative simplicity and their decentralized nature which allows for a large number of network users. In fact, Aloha-related concepts are still useful in modern wireless network protocols (for example, the 802.11 standards [6]). However, Aloha-based protocols are currently not fully adjusted to the distinctive features of the wireless medium, such as limited power and channel fading.

This paper attempts to find a middle ground between fully distributed mechanisms (such as Aloha) and centralized scheduling algorithms. Focusing on a single-cell uplink with throughput as the performance measure, we suggest a partially-distributed framework, which allows for users with different throughput requirements, while eliminating the need for on-line scheduling. Our approach consists of two phases. At the first (preliminary) phase, each mobile user is assigned a fixed data rate (or throughput). This assignment can be based on some fairness criterion, or can be the outcome of a price-based negotiation process (the rate assignment process is exogenous to our model). Subsequently, users adjust their transmission protocol in a *distributed* manner for obtaining the assigned rates. An equilibrium is obtained when all mobiles obtain their rates. We note that in some circumstances the required data rates for the users may be dictated by their application. In this case the required data rate may be regarded as self-enforced rather than explicitly allocated by the network. Our results apply to this case as well.

In this work, we restrict mobiles to a simple single-parameter transmission protocol, where the adjustable parameter stands for the individual's transmission probability. We assume that the transmission power is kept fixed in the time-frame considered (e.g., possibly through external power control), hence the transmission probability directly determines the station's average power investment. Allowing users to adjust their transmission probability in a distributed manner naturally adds an uncertainty factor to the system, as it is not apparent at which working point will the network operate. Equilibrium analysis plays a major role in understanding the network behavior. Our main interest in this paper is in characterizing the possible equilibrium points which may result from the user interaction, with emphasis on power-implications. An additional objective is to examine whether a reasonable update rule for the user transmission probabilities can be formulated in order to lead the network to efficient working points.

B. Related Literature

Distributed schemes in wireless networks have been well studied in the context of power control (e.g., in CDMA networks [7]). The research in this area also relates to scenarios in which the network users are self-optimizing [8], [9]. Recently, some papers have considered Aloha-like random access networks from a non-cooperative perspective ([10]–[12] and references therein), where users are allowed to modify their transmission parameters (e.g., the Aloha protocol parameters as in [12]) for their own best interest.

Of specific relevance to our work is a paper by Jin and Kesidis [10], which considers a shared collision channel with users who obtain fixed throughput assignments. A dynamic scheme where users adapt their transmission probabilities in order to satisfy their assigned throughput is proposed and studied in simulation. This work provides an analysis of an extended model with general channel characteristics, which in particular include capture and multipacket-reception channel models. In a related paper [13], we provide a detailed analysis of collision channels with channel state information (CSI).

C. Contribution and Paper Organization

This paper presents a detailed analysis of the fixed-rate equilibrium and further focuses on a dynamic mechanism which may lead the system to a power-efficient working point.

We start with a general wireless channel model, which may accommodate a variety of uplink reception models such as collision, capture [14], [15] and multipacket-reception [16] channels. When the assigned rates are feasible, we establish the existence of a uniformly best equilibrium point in terms of the power-investment. Consequently, we suggest a fully distributed mechanism which converges to this equilibrium.

Further equilibrium analysis is carried out for a more specific network model. We focus on a perfect capture channel where users can be divided to subgroups of equal strength. Under this specific model, we are able to characterize the feasible equilibrium region, bound the number of equilibrium points, and numerically compute all these points. In addition, we provide an upper-bound for the power investment at the best equilibrium. Finally, we show that the distributed mechanism suggested before is resilient to changes in the transmitters population, which obviously occur in wireless networks.

The structure of the paper is as follows. We first present the general model (Section II), and identify basic properties related to the average rate of each user. The existence of a best equilibrium is proven in Section III, followed by a mechanism which converges to this point (Section IV). The perfect capture model and the analysis thereof are given in Section V. Conclusion and further research directions are drawn in Section VI.

II. The Model and Preliminaries

Our model consists of a finite set of mobile users (or transmitters) $\mathcal{I} = \{1, \ldots, n\}$ who transmit to a common base station over a shared channel. The transmission power level for each user is pre-determined: it may be either fixed, or adjusted through some transmitter-receiver protocol. In either case, power level is not a decision variable in our model. Time is slotted, so that each transmission attempt takes place within slot boundaries that are common to all. To specify our model, we start with a description of a general multi-reception wireless channel (Section II-A), and list some central special cases thereof. In Section II-B, we characterize the user transmission regime, and define the notion of a fixed-rate equilibrium which is central in this paper.

A. Multipacket Reception Channels

We consider a shared wireless medium, where multiple transmissions arrive at the receiver at different power levels. In general, the per-slot throughput of each mobile increases with its own received power and decreases with the received power of other users, as we specify below.

We consider a single slot and omit the slot index for simplicity. Let I_i denote an indicator which equals one if user i transmits at a given slot and zero otherwise, and let $\mathbf{I} = (I_1, \ldots, I_n)$ be the vector of all user indicators. In addition, we use the notation $\mathbf{I}_{-i}$ for the indicator vector of all users but the ith one, so that $\mathbf{I} = (I_i, \mathbf{I}_{-i})$. Denote by $R_i(\mathbf{I})$ the average per-slot throughput for user i, given the set $\mathbf{I}_{-i}$ of other users that transmit simultaneously. Obviously, $R_i(I_i = 0, \mathbf{I}_{-i}) = 0$. Naturally, $R_i(\mathbf{I})$ should decrease as more users transmit, as we formalize below.

Assumption 1: For every vector $\mathbf{I}_{-k}$, $i \in \mathcal{I}$ and $k \neq i$,

$$R_i(I_k = 0, \mathbf{I}_{-k}) \geq R_i(I_k = 1, \mathbf{I}_{-k}). \tag{1}$$

Several common models comply with this assumption:

1) *Collision channel.* Simultaneous transmissions of two or more users result in a collision. Thus, $R_i(\mathbf{I}) = R > 0$ if $I_i = 1$ and $I_k = 0$ for every $k \neq i$, and $R_i(\mathbf{I}) = 0$ otherwise. This model was extensively used, for example in the study of Aloha-like protocols [5].

2) *Capture Channels.* The so-called capture effect takes place when a single (the strongest) user can be successfully received even in the presence of other simultaneous transmissions, provided that its power dominates the others' transmissions. This reception model is most common in WLAN receivers. Denote by w_i the received power of user i (in case of a transmission), and let $\mathbf{w} \triangleq (w_1, \ldots, w_n)$ be the vector of all received powers. A broadly studied capture model is based on the signal to interference plus noise ratio (SINR) (see, e.g., [15] and references therein). The SINR for user i is given by

$$\mathrm{SINR}_i(\mathbf{I}, \mathbf{w}) = \frac{I_i w_i}{\sum_{j \neq i} I_j w_j + \sigma_0}. \tag{2}$$

A transmission is successful if the SINR is large enough, namely if $\mathrm{SINR}_i > \beta > 1$ (where σ_0 is the ambient noise power). The per-slot throughput R_i is given by $R_i(\mathbf{I}) = \mathbb{E}^{\mathbf{w}}\{D_i 1\{\mathrm{SINR}_i(\mathbf{I},\mathbf{w}) > \beta\}\}$, where D_i is the data rate in case of a successful transmission, $1\{\cdot\}$ is a standard indicator function, and $\mathbb{E}^{\mathbf{w}}$ is the expectation operator, which averages all the stochastic elements at the network which determine the received powers. A more optimistic capture model assumes that the transmission with the strongest received power (say w_i) is received whenever $\frac{w_i}{w_k} > \beta$, for all $k \neq i$ (see, e.g., [14]). A special case of the latter model, which we focus on in Section V, is $\beta = 1$, known as the perfect capture model.
3) *Multipacket reception (CDMA).* In some wireless systems, such as cellular networks, multiple simultaneous receptions are possible. For example, in CDMA systems, the momentarily data rate of each user is given by $\log\big(1 + \mathrm{SINR}_i(\mathbf{I},\mathbf{w})\big)$, where the average throughput is obtained by averaging the last quantity, namely $R_i(\mathbf{I}) = \mathbb{E}^{\mathbf{w}}\big\{\log\big(1 + \mathrm{SINR}_i(\mathbf{I},\mathbf{w})\big)\big\}$.

B. User Model and Equilibrium

We associate with each user i a fixed data rate (or throughput) ρ_i (in bits per slot), assigned as an upper bound on the allowed throughput. This rate may be assigned (and policed) by the system, as a control and management tool, or determined by the user's application. In either case, the rate assignment procedure is exogenous to our model.

Each user sets its transmission schedule in a distributed manner for obtaining the allowed data rate ρ_i. In the paper we consider a stationary transmission schedule, determined by a single parameter p_i, which stands for the user's transmission probability in each slot. We assume that users always have packets to send, yet they may delay transmission to a later slot to accommodate their assigned throughput.

The transmission probability of each user affects the average throughput of the other users over the shared wireless channel. Let $r_i(\mathbf{p}) \equiv r_i(p_i, \mathbf{p}_{-i})$ denote user i's average throughput, as determined by the transmission probabilities $\mathbf{p} = (p_1, \ldots, p_n)$ of all users, namely $r_i(\mathbf{p}) = \sum_{\mathbf{I}\in\{0,1\}^n} \prod_{i=1}^n \big(p_i^{I_i}(1-p_i)^{1-I_i}\big) R_i(\mathbf{I})$. In view of $R_i(I_i = 0, \cdot) = 0$, this can be written as

$$r_i(\mathbf{p}) = p_i \bar{R}_i(\mathbf{p}_{-i}), \tag{3}$$

where $\bar{R}_i(\mathbf{p}_{-i})$ is the expected rate obtained in any transmission attempt of user i. The next lemma summarizes some basic properties of r_i. The proof is straightforward, with property (iii) following by Assumption 1.

Lemma 1: The average throughput function $r_i(p_i, \mathbf{p}_{-i})$ obeys the following properties:
(i) $r_i(p_i = 0, \mathbf{p}_{-i}) = 0$.
(ii) $r_i(\mathbf{p})$ is continuous in each of its arguments.
(iii) $r_i(p_i, \mathbf{p}_{-i})$ (when positive) strictly increases in p_i and decreases in p_j, $j \neq i$.

An *equilibrium* point is attained when the average throughput r_i of each user i is equal to the assigned data rate ρ_i. Formally, an equilibrium point is a vector of user probabilities $\mathbf{p} = (p_1, \ldots, p_n)$, which obeys the following set of equations:

$$r_i(\mathbf{p}) = \rho_i, \quad i \in \mathcal{I}. \tag{4}$$

We shall refer to these equations as the *equilibrium equations*. Apparently, each user is interested in an equilibrium point where its transmission probability is minimal, as the quantity is proportional to its respective power investment.

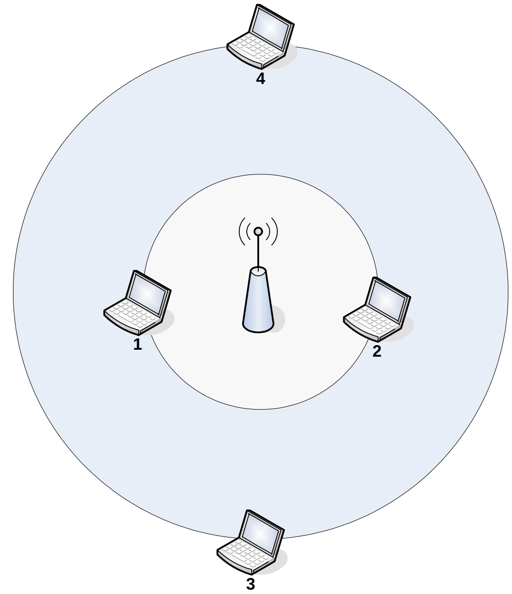

Fig. 1. The four node network. Stations 1 and 2 are closer to the base station, thus not interrupted by transmissions from Stations 3 and 4.

III. Existence of a Best Equilibrium

A significant factor in our model is that users are allowed to adjust their transmission probability in a distributed manner. The clear advantage of this framework lies in the substantial reduction of management overhead, compared, for example, to detailed scheduling mechanisms. At the same time, however, an uncertainty factor is added to the system, in the form of the user choices. Equilibrium analysis plays a major role in this context. We are interested in characterizing the possible equilibrium points in which all users satisfy their rate requirements, their number and quality (in terms of power investment). Consequently, we wish to examine whether the system dynamics (namely, the iterative process by which the user determines its transmission probability) can be channeled in some reasonable manner to lead the network to power-efficient working points.

Under our general assumptions on the multiuser wireless channel, we provide in this section a partial answer to the issues raised above, by showing the existence of a uniformly best equilibrium point. Several equilibrium aspects can be analyzed only while considering a more specific channel, as we do in Section V.

A motivating numerical example is presented below.
Example 1. Consider the network depicted in Figure 1. As users 1 and 2 are closer to the base station, their received

power is larger than that of users 3 and 4. As a special case of the SINR-based capture rule (see Section II-A), we assume that $\beta \approx 1$, thus the transmissions of users 1 and 2 are not interrupted by transmissions from users 3 and 4. Consequently, user 1's transmission fails only if user 2 transmits simultaneously (and vice versa). User 3, as well as user 4, require that no other user will simultaneously transmit for proper reception of their transmissions (note that the same relations can be obtained in a different network scenario where users 1 and 2 dominate users 3 and 4 as a result of transmitting at a higher power level). For simplicity, we assume that a successful transmission is of one unit. The equilibrium equations (4) for this specific case are given by

$$\begin{aligned}
&p_1(1-p_2) = \rho_1,\\
&p_2(1-p_1) = \rho_2,\\
&p_3(1-p_1)(1-p_2)(1-p_4) = \rho_3,\\
&p_4(1-p_1)(1-p_2)(1-p_3) = \rho_4. \qquad (5)
\end{aligned}$$

Let the users' required data rates be the following: $\rho_1 = \rho_2 = 0.23$; $\rho_3 = \rho_4 = 0.05$. As we shall prove in Section V-A, there are exactly two different transmission probabilities for users 1 and 2 which yield their required rates, namely $p_1 = p_2 = 0.36$ or $p_1 = p_2 = 0.64$. In the latter case (irrespectively of user 3 and 4 choices), a *partial* equilibrium is obtained, as these users satisfy their own rates, while the data rates of users 3 and 4 cannot be sustained (i.e., there is no assignment of p_3 and p_4 such that the third and fourth equations in (5) are satisfied). However, if users 1 and 2 use the lower probability of 0.36 (which is better for them in terms of power investment), two equilibria are obtained: $(0.36, 0.36, 0.14, 0.14)$ and $(0.36, 0.36, 0.86, 0.86)$. Note that the first of the two equilibria is *best for all users*. Note further that if users 1 and 2 indeed choose the lower transmission probability, the starvation effect shown above can be avoided.

The example above clearly demonstrates that multiple equilibria are possible, and that starvation of some users might occur even if a feasible equilibrium point does exist. These observations raise the following basic questions:

1) Is there always an equilibrium point which is preferable to *all* users?
2) If there exists such an equilibrium, could it be reached in a distributed manner?

A positive answer to these questions can avoid power-expensive equilibria and unnecessary user starvation, as all users should be willing to adopt a mechanism which converges to an equilibrium point that is optimal from their point of view.

A "best equilibrium" is an equilibrium point where *all* users transmit with minimal probability, compared to their transmission probabilities at any other equilibrium. As such, the best equilibrium is power-superior to all other equilibria, uniformly over all users. We assert below that when an equilibrium point exists, one of the equilibrium points is a best equilibrium.

Theorem 1: In case that an equilibrium point defined by (4) exists, one equilibrium is best for all users. That is, there exists an equilibrium point $\mathbf{p}^*$ such that $\mathbf{p}^* \leq \tilde{\mathbf{p}}$ for any other equilibrium point $\tilde{\mathbf{p}}$.

Proof: The idea of the proof is to apply Tarski's fixed point theorem (see, e.g., [17]) on an increasing function $h : [0,1]^n \mapsto [0,1]^n$, defined so that every equilibrium (4) is a fixed point of that function.

For each $\mathbf{p} = (p_1, \ldots, p_n)$, consider the following best-response mapping,

$$\mathbf{p} = (p_1, \ldots, p_n) \in [0,1]^n \mapsto (h_1(\mathbf{p}), \ldots, h_n(\mathbf{p})) \in [0,1]^n,$$

where

$$h_i(\mathbf{p}) = \begin{cases} p_i^* & \text{if } \exists p_i^* \text{ so that } r_i(p_i^*, \mathbf{p_{-i}}) = \rho_i \\ 1 & \text{otherwise,} \end{cases} \qquad (6)$$

Note that this mapping is uniquely defined by the strict monotonicity of r_i in p_i (Lemma 1). It is evident that every equilibrium point $\mathbf{p}$ is such that $\mathbf{p} = \big(h_1(\mathbf{p}), \ldots, h_n(\mathbf{p})\big)$, where h_i is given by (6). Yet, additional "artificial" equilibria, such as $p_i = 1$ for every $i \in \mathcal{I}$ may arise by this definition. Let $h(\mathbf{p}) = \big(h_1(\mathbf{p}), \ldots, h_n(\mathbf{p})\big)$. The following monotonicity property easily follows.

Lemma 2: $h(\mathbf{p})$ is an increasing function, that is $h_i(\mathbf{p})$ is (weakly) increasing in p_j for all i and j.

Proof: Let $\tilde{\mathbf{p}}$ and $\mathbf{p}$ be two probability vectors such that $\tilde{\mathbf{p}} \geq \mathbf{p}$. It is required to show that $h_i(\tilde{\mathbf{p}}) \geq h_i(\mathbf{p})$. If $h_i(\mathbf{p}) = 1$ then obviously $h_i(\tilde{\mathbf{p}}) = 1$, as r_i decreases in $\mathbf{p}_{-i}$ by Lemma 1. Next, consider the case where $h_i(\mathbf{p}) < 1$. If $h_i(\tilde{\mathbf{p}}) = 1$ we are done. Otherwise,

$$r_i(h_i(\tilde{\mathbf{p}}), \tilde{\mathbf{p}}_{-\mathbf{i}}) = r_i(h_i(\mathbf{p}), \mathbf{p}_{-\mathbf{i}}) = \rho_i. \qquad (7)$$

The required result follows directly from (7) by recalling that r_i decreases with $\mathbf{p_{-i}}$ and increases with p_i. $\square$

We can now apply Tarski's fixed point theorem to obtain the required result. Define $S \stackrel{\triangle}{=} [0,1]^n$. Note that S is a complete lattice (see, e.g., [17] for definition). $h(\mathbf{p})$ is an increasing function from S to S (by Lemma 2). Then by Tarski's fixed point theorem the set of fixed points of h is non-empty. Moreover the supremum and the infimum of the set are fixed points themselves. The supremum fixed point is always $p_i = 1$ for every i, which is not an equilibrium point in the sense of (4), as required throughput demands are not met. In case there is no equilibrium in the original game, the fixed points obtained by Tarski's theorem are all artificial (i.e., include users with $p_i = 1$ who cannot obtain their throughput demands). Yet, when throughput demands are feasible, Tarski's theorem implies that an infimum equilibrium point exists. $\square$

We emphasize that Theorem 1 is valid under our general assumptions on the rate function r_i. The issue of the *existence* of an equilibrium, however, requires a specific study for each network under consideration. Obviously, if the overall data rates of the users are too high there cannot be an equilibrium point, since the network naturally has limited traffic capacity. Explicit network features, such as the reception model, the uplink channel quality (or gain) of each user and the user topology, determine the capacity of the network, hence the existence of an equilibrium for given data rates ρ_i. We address the existence issue for the perfect capture model in Section V.

We conclude this section with a comment on sub/supermodular (or just S-modular) games [17], [18] and their relation to our model. Roughly, S-modular games is a subclass of non-cooperative games, in which the user (or player) utility functions obey certain monotonicity properties. These properties lead to strong results regarding the existence and structure of Nash equilibria, along with some convergence properties thereof. In particular, a best and worst Nash equilibrium always exist in these games, and synchronous greedy mechanisms convergence to an equilibrium under certain conditions (see [17], [18]).

In our model, users are not utility maximizers, and therefore the general framework is not that of a non-cooperative game. We do use monotonicity properties of the rate functions (Lemma 1) for proving the existence of a best equilibrium, by applying similar tools (such as Tarski's fixed point theorem) to the ones used in the general theory of S-Modular games. We could thus conjecture that, as in S-modular games, there exists a *worst* equilibrium in our model, however we show through a simple example (Example 2, Section V-A) that this is not case. Moreover, the mechanism we suggest next for converging to the best equilibrium is fully asynchronous, a case not explicitly considered in the S-modular games literature.

IV. Convergence to the Best Equilibrium

We have shown so far that when an equilibrium point exists under our multipacket reception channel model, there is always a best equilibrium point, which all users would prefer, as their power investment is minimal. This leads us to find a mechanism which converges to the best equilibrium.

The distributed mechanism we suggest is informally described as follows. *Each user updates its transmission probability from time to time, by matching its average throughput r_i to the required rate ρ_i.* Formally, let the update time-slots of each user i be given by an increasing sequence $\{t_i^k\}$, $k = 1, 2, 3, \ldots$. At time slot t_i^k, user i sets its transmission probability as follows.

$$p_i(t_i^k) := \frac{\rho_i}{\bar{R}_i(\mathbf{p}_{-i}(t_i^k - 1))}, \tag{8}$$

where $\bar{R}_i$ is defined in (3), and $\mathbf{p}_{-i}(t_i^k - 1)$ is the transmission probability vector of all users (but the ith one) just before time slot t_i^k. In case that $\frac{\rho_i}{\bar{R}_i(\mathbf{p}_{-i}(t_i^k-1))}$ exceeds the value of one, $p_i(t_i^k)$ can be set to an arbitrary probability. Note that (8) is in fact the best-response (a term often used in game-theoretic context [19]) of the user to a given network situation, in the sense that the maximal allowed rate is obtained. We emphasize that the suggested mechanism does not require any synchronization between users, as each user independently chooses the update times t_i^k. The motivation for using the above rule follows directly from (3) and (4), noting that (8) sets (whenever possible) the current user throughput at ρ_i which is the required throughput at equilibrium. For the convergence analysis of the mechanism we require the following set of assumptions.

Assumption 2:
(i) The user population is fixed.
(ii) Transmission probabilities are initialized to zero ("slow start").
(iii) Users repeatedly update their transmission probabilities (i.e., $t_i^k \to \infty$ as $k \to \infty$).
(iv) The effective quantity $\bar{R}_i(\mathbf{p}_{-i}(t_i^k - 1))$ is perfectly estimated by the user before each update.
Our convergence result is summarized below.

Theorem 2: Assume an equilibrium point exists. Then under Assumption 2, the distributed mechanism (8) asymptotically converges to the best equilibrium.

Proof: The proof of the above result relies on showing that the vector of user probabilities $\mathbf{p}$ monotonously increases until convergence. A full proof is given in the appendix.

We briefly list here some important considerations regarding of the presented mechanism.
1) The slow start requirement (Assumption 2(ii)) is essential for preventing excessive transmissions which lead to suboptimal equilibria.
2) The quantity $\bar{R}_i(\mathbf{p}_{-i}(t_i^k - 1))$ required in (8), can be estimated by keeping track of the history of previous transmissions (note that the particular value of each p_k, $k \neq i$ is not essential here).
3) Assumption 2(iv) entails the notion of a quasi-static system, in which each user responses to the steady state reached after preceding user updates. This assumption approximates a natural scenario where users update their transmission probabilities at much slower time-scales than their respective transmission rates.
4) The convergence results obtained in the section would still hold for a relaxed variation of (8), given by $p_i(t_i^k) := \beta_i \frac{\rho_i}{\bar{R}_i(\mathbf{p}_{-i}(t_i^k-1))} + (1 - \beta_i)p_i(t_i^k - 1)$, $0 < \beta_i \leq 1$. This update rule can be more robust against inaccuracies in the estimation of $\bar{R}_i(\mathbf{p}_{-i}(t_i^k-1))$, perhaps at the expense of slower convergence to the desired equilibrium.

Our convergence result is obviously idealized and should be supplemented with further analysis of the effect of possible deviations from the model and possible remedies. In case that a worse equilibrium point does occur, users can reset their probabilities and restart the mechanism (8) for converging to the best equilibrium. This procedure resembles the basic ideas behind TCP protocols. The exact schemes for detecting suboptimal equilibria, and consequently directing the network to the best equilibrium are beyond the scope of the present paper.

V. The Multi-Ring Network

In this section we focus on a specific channel model, under which we extend the scope of the equilibrium analysis, addressing issues such as existence and number of equilibria, the overall power investment, and stability.

We consider an uplink model, where mobiles are divided into subgroups $\mathcal{I}_1, \ldots, \mathcal{I}_M$ of approximately equal power strength. We assume that these subgroups are ordered in decreasing strength. We further assume perfect capture (see Section II-A), meaning that a transmission is successful if no other user with equal or larger power attempts transmission. Accordingly, the transmission of a user $i \in \mathcal{I}_m$ is successful if no other user $j \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_m$ transmits simultaneously. The average data rate in a successful transmission is given by the constant D_i for every $i \in \mathcal{I}$.

The above model fits the following network scenarios:

- *Discrete power levels.* The transmission power of each mobile is chosen from a finite set of power levels $W_1 > W_2 > \cdots > W_M$. The above capture model is valid when mobiles are located at similar distances from the base station.
- *Multi-ring topology.* Mobiles with equal transmission powers are positioned in a multi-ring topology, as in Figure 1, where each ring contains mobiles with approximately equal distance from the base station.

For convenience, we shall refer to our network as a multi-ring network, where a subset $\mathcal{I}_m$ will be considered as the mth ring. The term M-ring network is used for a network with M rings. The content of this section is as follows. In Section V-A we provide a detailed equilibrium analysis for the multi-ring network, investigating the possible number of equilibrium points and their structure, with focus on the best equilibrium point. In light of the distributed mechanism presented in Section IV, we examine whether the mechanism re-converges to the best equilibrium point when changes in the user population occur (Section V-B).

A. Equilibrium Analysis

Let us start by revisiting the two-ring, four user network configuration, depicted in Figure 1. Consider the following numerical example:

Example 2. We use the same network parameters as in Example 1 (Section III), except for the user data rates which are now given by $\rho_1 = \rho_2 = 0.23$; $\rho_3 = \rho_4 = 0.02$. In this case there are exactly four equilibria: $\mathbf{p^{(1)}} = (0.36, 0.36, 0.05, 0.05)$, $\mathbf{p^{(2)}} = (0.36, 0.36, 0.95, 0.95)$, $\mathbf{p^{(3)}} = (0.64, 0.64, 0.19, 0.19)$, $\mathbf{p^{(4)}} = (0.64, 0.64, 0.81, 0.81)$.

While a best equilibrium exists ($\mathbf{p^{(1)}}$), Example 2 reveals that, generally, there is no common *worst* equilibrium for all users ($\mathbf{p^{(2)}}$ is worst for users 3 and 4, while $\mathbf{p^{(3)}}$ and $\mathbf{p^{(4)}}$ are worst for users 1 and 2). As to the possible number of equilibria, we show below that four is indeed the maximal number of equilibrium points for any two-ring configuration. More generally, we prove that the maximal number of equilibria for an M-ring network is given by 2^M. Our ability to bound the maximal the number of equilibrium points is a significant step in calculating all equilibria (as we demonstrate in the sequel), thus predicting the overall network behavior.

Our equilibrium analysis relies on first examining the collision channel model (see Section II-A), which is a special case of the multi-ring network with $M = 1$.

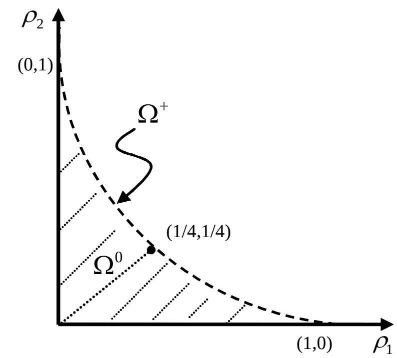

Fig. 2. The set of feasible throughput vectors Ω for a two user network with $D_i = 1$, $i = 1, 2$. Ω^0 is the interior of Ω and Ω^+ is the pareto-optimal set of rates.

1) Collision Channel Analysis ($M = 1$): Properties of the equilibrium and feasible throughputs for a collision channel were studied in detail in [13] (where more general individual channel properties were considered). We repeat here some results from [13] that will be useful for the M-ring model.

Recall that in a collision channel simultaneous transmissions of two or more users result in a collision and loss of all data. Thus, the equilibrium equations for this capture channel are given by

$$D_i p_i \prod_{j \neq i} (1 - p_j) = \rho_i, \quad i \in \mathcal{I}. \tag{9}$$

Obviously, if the overall throughput demands of the users are too high there cannot be an equilibrium point, since the network naturally has limited traffic capacity. Denote by $\rho = (\rho_1, \ldots, \rho_n)$ the data rate vector of all users, and let Ω be the set of *feasible* vectors ρ, for which there exists at least one equilibrium point (equivalently, for which there exists a feasible solution to (9)). Figure 2 illustrates the set of feasible throughput vectors for a simple two-user case, with $D_i = 1$, $i = 1, 2$.

We do not specify here the exact structure of Ω; details are given in [13]. It is easily verified that Ω is a closed set with nonempty interior. In addition, we provide in [13] a sufficient condition for the feasibility of ρ, given by $\sum_i \frac{\rho_i}{D_i} \leq e^{-1}$. We specify below the number of equilibrium points for any throughput demand vector $\rho = (\rho_1, \ldots, \rho_n)$ in the interior of Ω.

Proposition 3 ([13], Theorem 3): Consider a collision channel, and let Ω be the set of feasible data rate vectors $\rho = (\rho_1, \ldots, \rho_n)$. Denote by Ω^0 the interior of Ω. Then for each $\rho \in \Omega^0$ there exist exactly two equilibria.

Proof: (outline) We sketch the proof idea for completeness. The idea is to reduce the equation set (9) to a single scalar equation in a single variable p_i, for some arbitrarily chosen user i, and investigate this scalar equation.

Let $y_i = \frac{\rho_i}{D_i}$, and define $a_{ji} \triangleq y_j / y_i$. It can be seen that

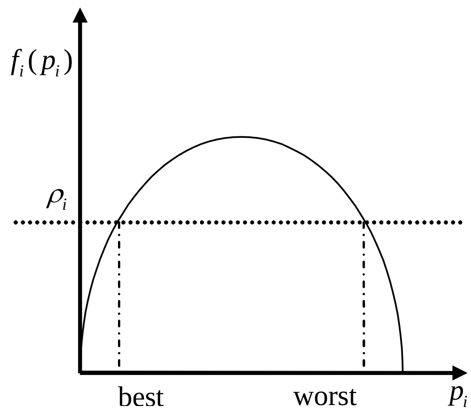

Fig. 3. The scalar function $f_i(p_i)$ used in the proof of Proposition 3. The values of $f_i(p_i)$ equal the throughput which user i will obtain, assuming that the required throughput ratios between user i and other users are kept.

the following relation holds in every equilibrium point and for every $i, j \in \mathcal{I}$.

$$p_j = \frac{a_{ji} p_i}{1 - p_i + a_{ji} p_i} \stackrel{\triangle}{=} p_j(p_i). \tag{10}$$

This relation is immediately obtained by dividing the equilibrium equation (9) of the ith user by the equation of the jth one. Substituting (10) in the i-th equilibrium equation, we obtain the following scalar equation for p_i:

$$f_i(p_i) = D_i p_i \prod_{j \neq i} (1 - p_j(p_i)) = \rho_i. \tag{11}$$

Unimodality of $f_i(p_i)$ will establish the required result, as a unimodal function can obtain a given value ρ_i at most twice (see Figure 3). The required unimodality is established in [13, Theorem 3]. □

Certain computational properties of an equilibrium point may be directly observed from the proof of Proposition 3. It can be seen that verifying the existence of an equilibrium (for given data rates ρ) is computationally equivalent to finding the maximum of a scalar unimodal function. Similarly, the equilibrium points themselves are computed by finding the zeros of the function $f_i(p_i) - \rho_i$. This suggests that the computation of the equilibrium can be efficiently accomplished by standard search techniques, such as the bisection method or the golden section search method (see, e.g., [20]).

2) The General Case ($M \geq 1$): For a concrete study of the existence and number of equilibria, we introduce the notion of a partial equilibrium. Let $\mathcal{I}_{1:m} \stackrel{\triangle}{=} \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_m$ denote the users in the inner m rings.

Definition 5.1 (Partial Equilibrium): A partial equilibrium for $\mathcal{I}_{1:m}$, denoted $\mathbf{p}^m \in [0,1]^{|\mathcal{I}_{1:m}|}$, is a vector of user probabilities such that the equilibrium equations (4) are met for every $i \in \mathcal{I}_{1:m}$.

For any partial equilibrium $\mathbf{p}^m$, we denote by $Q_m(\mathbf{p}^m) \stackrel{\triangle}{=} \prod_{i \in \mathcal{I}_{1:m}} (1 - p_i)$ the idle probability of users belonging to the m inner rings.

Based on Proposition 3, we are able to bound the maximal number of equilibrium points for any number of rings $M \geq 1$. In addition, we can characterize the feasible set of data rates $\rho^{m+1} = (\rho_i)_{i \in \mathcal{I}_{m+1}}$ for the $(m+1)$-ring users as a function of the partial equilibrium probabilities $\mathbf{p}^m$ of users at inner rings.

Theorem 4: Consider a ring network with $M \geq 1$. Then
(i) The maximal number of equilibrium points is 2^M.
(ii) Given a partial equilibrium $\mathbf{p}^m$, a throughput vector ρ^{m+1} for the $(m+1)$st ring is feasible if an only if $Q_m^{-1}(\mathbf{p}^m)\rho^{m+1}$ is feasible in a collision channel with $|\mathcal{I}_{m+1}|$ users.

Proof: Note that given a partial equilibrium $\mathbf{p}^m$, the $(m+1)$st ring can be considered as a collision channel, with a (constant) idle probability $Q_m(\mathbf{p}^m)$ multiplying the left hand side of the equilibrium equations (9), namely

$$Q_m(\mathbf{p}^m) D_i p_i \prod_{j \in \mathcal{I}_{m+1} \setminus i} (1 - p_j) = \rho_i, \quad i \in \mathcal{I}_{m+1}. \tag{12}$$

Choosing an arbitrary user $i \in \mathcal{I}_{m+1}$, we can reduce the equilibrium equations to a similar scalar equation to the one used in the proof of Proposition 3, where the only difference is that the left-hand side is now multiplied by the constant $Q_m(\mathbf{p}^m)$, namely

$$\tilde{f}_i(p_i) = Q_m(\mathbf{p}^m) D_i p_i \prod_{j \in \mathcal{I}_{m+1} \setminus i} (1 - p_j(p_i)) = \rho_i, \tag{13}$$

where $p_j(p_i)$ is given in (10). We are now ready to prove the theorem's claims: (i) The constant factor in (13) would obviously not change the unimodality property of the function (11), which is used for proving the two-equilibria property, hence $\tilde{f}_i(p_i)$ is unimodal in p_i. Thus, each partial equilibrium $\mathbf{p}^m$ creates at most two equilibrium values for the $(m+1)$st ring, leading to the required result. (ii) It can be easily observed that if $Q_m^{-1}(\mathbf{p}^m)\rho^{m+1}$ obtains a feasible solution for (11), ρ^{m+1} obtains a feasible solution for (13), and vice versa. □

A smaller number of equilibrium points is obtained in cases such as the one described in Example 1, where strong users might prevent weaker users from obtaining their required data rates. We briefly discuss how to calculate all equilibrium points. It turns out that all equilibria can be calculated by an iterative procedure which is based on our ability to compute the equilibrium point of a collision channel. The first step in the procedure is to calculate the equilibrium probabilities for $i \in \mathcal{I}_1$, as these users are not affected by transmissions from outer rings. For each partial-equilibrium $\mathbf{p}^1$ we proceed to calculate the equilibrium probabilities for $\mathcal{I}_2$ as if this ring is an independent collision channel, where an idle probability $Q_1(\mathbf{p}^1)$ multiplies the left-hand side of (9). This procedure carries over to outer rings, creating a tree-like structure of equilibria.

We conclude our analysis by addressing the quality of the best equilibrium in terms of the total transmission probability.

Corollary 1: Let $\mathbf{p}$ be the best equilibrium for the multi-ring network. Then $\sum_{i \in \mathcal{I}_m} p_i < 1$ for every $m = 1, \ldots, M$.

Proof: In [13, Theorem 5] we established that the best equilibrium $\mathbf{p}$ in a collision channel satisfies $\sum_i p_i < 1$. By Theorem 4(ii) we can treat each ring as a collision channel, hence the above inequality must hold separately at every ring. □

The significance of the above theorem is that the overall transmission power at the best equilibrium is bounded by a known quantity. For example, assuming that mobiles at the mth ring transmit at the same power level of W_m, the overall power investment (per slot) is bounded by $\sum_{m=1}^{M} W_m$. We note that this result is also used in the proof of Theorem 5 below

B. Resilience to Changing User Population

Convergence to the best equilibrium has been established under the assumption of a fixed user population (Section IV, Theorem 2). However, the user population (and the assigned data rates) change over time. Hence, it is important to study the system dynamics when users join or leave the network, as is often the case in wireless networks. For our analysis, we assume that the network is at equilibrium, i.e., the required throughput demands are met for the present users, when new users join or leave. The next result shows that the network is resilient to a change in user population, in the sense that the mechanism (8) re-converges to the best equilibrium of the presently active users.

Theorem 5 (Changing user population): Consider a network which is at its best equilibrium, and the next two scenarios: (i) some users join the network (not necessarily at the same time slot). (ii) some users leave the network (not necessarily at the same time slot). Then under Assumptions 2(iii) and 2(iv), the mechanism (8) will asymptotically converge to best equilibrium of the active users.

The case of joining users essentially follows from Theorem 2, as the initial transmission probabilities are below the best equilibrium probabilities, thus are increased by the user updates. However, when users leave the network, present users are required to lower their transmission probabilities for obtaining the best equilibrium. Hence, in the latter case, a different proof method is required. A detailed proof of Theorem 5 is thus provided in the appendix. The case where some users join and some abandon before convergence is more involved, and requires further investigation.

VI. Conclusion

This paper suggests a framework for MAC management in wireless networks, in which the system assigns the user performance levels, while users are responsible for sustaining the assigned quantities in a distributed manner. We focused here on a general uplink model where users are allocated an allowed data rate, which they wish to maintain over time. Restricting users to a single-parameter transmission protocol, we showed that while multiple equilibrium points exist in general, one of these equilibria is best for all users. To avoid sub-optimal working points, we suggested a distributed mechanism that converges to the uniformly best equilibrium point. Concentrating on a perfect capture channel, we characterized the power investment at the best equilibrium, and showed that this equilibrium is stable.

The framework and results of this paper may be extended in several ways. First, additional specific channel models may be investigated. A central issue in any specific model would be how to detect suboptimal equilibria and lead the network to the best equilibrium point. Additionally, one may consider more complicated transmission protocols, which take into account channel feedbacks (such as an indication for a successful transmission). A central question is whether the system benefits from the use of more complex protocols by the individuals. An additional degree of freedom which we did not consider here, is to allow users to autonomously modify their transmission power, in conjunction with adjusting the transmission schedule.

At a somewhat higher level, a challenging direction would be to extend the scope of the two-phase rate-based framework suggested here to multi-hop networks, while considering additional performance criteria (such as delay).

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APPENDIX

For the proofs in this section we use the following notations. Let $\{t^k\} = \left\{\{t_1^k\} \cup \{t_1^k\} \cup \ldots \{t_n^k\}\right\},\ k = 1, 2, \ldots$ Note that at each t^k at least a single user updates its transmission probability. We shall use the notation p_i^k for the transmission probability of user i at time t^k (similarly, $\mathbf{p}^k$ is the transmission probability vector at time t^k), with the convention of $p_i^0 = 0$ for every user i.

Proof of Theorem 2: For the proof of the theorem we require the next lemma.

Lemma 3: The sequence $\mathbf{p}^k$ is increasing.

Proof: The result follows by induction. Obviously, $\mathbf{0} = \mathbf{p}^0 \leq \mathbf{p}^1$. Assume that $\mathbf{p}^0 \leq \mathbf{p}^1 \leq \ldots \mathbf{p}^{k-1}$. We next show that $\mathbf{p}^{k-1} \leq \mathbf{p}^k$. Denote by I^k the set of users who update their probabilities at time k (so that $p_i^{k-1} = p_i^k\ \forall i \notin I^k$). For each $i \in I^k$, let $k_i < k$ be the last time epoch at which user i updated its probability. Note that

$$r_i(p_i^{k-1}, \mathbf{p}_{-i}^{k_i-1}) = r_i(p_i^k, \mathbf{p}_{-i}^{k-1}) = \rho_i. \tag{14}$$

Since $r_i(\mathbf{p}) = p_i \prod_{j \neq i}(1 - p_j)$ is decreasing in $\mathbf{p}_{-i}$ and, by assumption, $\mathbf{p}_{-i}^{k_i-1} \leq \mathbf{p}_{-i}^{k-1}$, it follows that $p_i^{k-1} \leq p_i^k$ (as r_i is increasing in p_i). □

It follows from the above lemma that either some component of $\mathbf{p}$ must exceed 1 at some iteration, or else $\mathbf{p}$ approaches a limit, say $\mathbf{p}^*$, and in this limit the equilibrium equations (4) are obviously satisfied (by continuity), i.e., it is an equilibrium point.

To conclude the proof, we now turn to show that if $\tilde{\mathbf{p}}$ is (another) equilibrium point, then $\mathbf{p}^* \leq \tilde{\mathbf{p}}$. To see this, we apply a similar induction as that of Lemma 3, and also use the notations thereof. Obviously $\mathbf{0} = \mathbf{p}^0 \leq \tilde{\mathbf{p}}$. Assume $\mathbf{p}^0 \leq \mathbf{p}^1 \leq \cdots \leq \mathbf{p}^{k-1} \leq \tilde{\mathbf{p}}$. Noting that $r_i(p_i^k, \mathbf{p}_{-i}^{k_i-1}) = r_i(\tilde{p}_i, \tilde{\mathbf{p}}_{-i}) = \rho_i$ and $\mathbf{p}_{-i}^{k_i-1} \leq \tilde{\mathbf{p}}_{-i}$ for every $i \in \mathcal{I}$, it follows that $p_i^k \leq \tilde{p}_i$, i.e., $\mathbf{p}^k \leq \tilde{\mathbf{p}}$. This argument also shows that if some component of $\mathbf{p}^k$ exceeds 1 for some k, then there is no equilibrium point (i.e., the set of the assigned throughputs $\{\rho_i\}$ is infeasible). □

Proof of Theorem 5: Convergence of case (i) (joining users) follows directly from the convergence property of the mechanism itself (Theorem 2), as joining users can be regarded as users who have been present at the network, yet decide to update their probabilities at late times.

For case (ii), we restrict the discussion to a *single* departure, for simplicity of exposition. Results for multiple departures are obtained through the same arguments. We start our analysis with a lemma which compares the best equilibria for two throughput vectors.

Lemma 4: Let ρ and $\tilde{\rho}$ be two throughput demand vectors such that $\tilde{\rho} \geq \rho$ (component-wise), and let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ denote the respective best equilibria. Then $\tilde{\mathbf{p}} \geq \mathbf{p}$. Consequently, fixing some ρ for n users, the best equilibrium transmission probabilities are lower (component-wise) with $n - 1$ users present, in comparison to the best equilibrium transmission probabilities with n users present.

Proof: For the proof, we track the distributed mechanism for the case of parallel updates (where t_i^k does not depend on i), which are guaranteed to converge to an equilibrium point by Theorem 2. We next show that $\tilde{\mathbf{p}}^{\mathbf{k}} \geq \mathbf{p}^k$ for every k, thus also at the limit. Note that since $r_i(\tilde{p}_i^1, \mathbf{0}) = \tilde{\rho}_i \geq r_i(p_i^1, \mathbf{0}) = \rho_i$, then by the monotonicity of r_i, $\tilde{p}_i^1 \geq p_i^1$ for every i. At the next iteration, $r_i(\tilde{p}_i^2, \tilde{\mathbf{p}}_{-i}^1) = \tilde{\rho}_i \geq r_i(p_i^1, \mathbf{p}_{-i}^1) = \rho_i$. Since $\tilde{\mathbf{p}}_{-i}^1 \geq \mathbf{p}_{-i}^1$, it follows that $\tilde{p}_i^2 \geq p_i^2$ for every i. The same argument carries over to subsequent iteration, thus it is valid also at the limit. The case of $(n-1)$ users is obtained as a special case of the above, by setting $\rho_n = 0$. □

We are now ready to prove convergence for the case of a leaving user. The impact of an abandoning user (say the nth one) is equivalent to setting $p_n = 0$. Let $\hat{\mathbf{p}}$ denote the initial probability vector, representing the best equilibrium when n users were present and let $\mathbf{p}^0$ denote the same vector, except that $p_n = 0$. For the proof of the claim, we require the next two lemmas.

Lemma 5: In case of an abandoning user, the sequence $\mathbf{p}^k$ is decreasing.

Proof: Denote by I^1 the subset of users who update their probabilities at $k = 1$. For every $i \in I^1$, since $r_i(p_i^1, \mathbf{p}_{-i}^0) = r_i(p_i^0, \hat{\mathbf{p}}_{-i}) = \rho_i$, it follows by the monotonicity of r_i that $p_i^1 \leq p_i^0$ Thus overall, $\mathbf{p}^1 \leq \mathbf{p}^0$. The result of the lemma follows by proceeding similarly in subsequent iterations (see a similar proof idea in Theorem 2). □

Lemma 6: The sequence $\mathbf{p}^k$ is bounded below by the best equilibrium of the $n - 1$ users.

Proof: Denote by $\mathbf{p}^*$ the best equilibrium with $n - 1$ users. Then by Lemma 4 $\mathbf{p}^0 \geq \mathbf{p}^*$. Denote by I^1 the subset of users who update their probabilities at $k = 1$. For these users we have $r_i(p_i^1, \mathbf{p}_{-i}^0) = r_i(p_i^*, \mathbf{p}^*{}_{-i})$. Since $\mathbf{p}_{-i}^0 \geq \mathbf{p}^*{}_{-i}$ it follows that $p_i^1 \geq p_i^*$ for every i. This argument may be carried over to subsequent iterations ($\mathbf{p}_{-i}^k \geq \mathbf{p}^*{}_{-i}$ for every k) and the result follows. □

An immediate consequence of the last two lemmas is that the mechanism reobtains an equilibrium in the case that a user leaves the network. We now show that the mechanism converges to the best equilibrium. Recall by Corollary 1 that the best equilibrium obeys $\sum_{i \in \mathcal{I}_m} p_i < 1$ for every $m = 1, \ldots, M$. This is true, in particular, for the equilibrium point with n users. Since the sequence $\mathbf{p}^k$ decreases due to the abandonment of a single user, it follows that $\sum_{i \in \mathcal{I}_m} p_i^k < 1$ for every k. Accordingly, the convergence of the sequence (guaranteed by the above two lemmas) must be to the best equilibrium of the $(n - 1)$ users. □