Strategic Posting Times over a Shared Publication Medium

Eitan Altman
INRIA
2004 Route des Lucioles
06902 Sophia-Antipolis Cedex, France
Email: altman@sophia.inrial.fr

Nahum Shimkin
Department of Electrical Engineering
Technion—Israel Institute of Technology
Haifa 32000, Israel
Email: shimkin@ee.technion.ac.il

***Abstract*—We consider a game of timing between advertisers, or other content creators, who compete for position and exposure over a shared publication medium such as an on-line classified list. Posted items (such as ads, messages, multimedia, or comments) are ordered according to their posting times, with recent posts displayed at the top positions. The effectiveness of each item depends on its current display position, as well as on a time-dependent exposure function which represents the collective exposure of the medium. It is assumed that each of a Poisson-distributed number of advertiser may choose the posting time of his item within a finite time interval, with the goal of maximizing the total exposure of this item. We formulate the problem as a non-cooperative game between advertisers, and analyze the Nash equilibrium profile of this game in terms of existence, uniqueness, computation and efficiency. Explicit expressions are derived for the case where the relative importance of the posting positions are geometrically decreasing.**

I. Introduction

Consider an online media site, to which each of several users can post a media item such a text message, picture, or video clip. We shall henceforth refer to these users as *players*. Each individual player can choose the time to post his or her item. Posted items are displayed as an ordered list, with the more recently-posted items displayed at the top (and better) positions, and gradually scrolled to lower positions (and eventually removed) as newer items are posted. The motivation for this model comes from various chronologically-arranged lists on the Web, which include online classified ad sites (such as Craigslist and many others), comment and response posting on news websites, Internet discussion forums, online book and product customer reviews, per-equity commentary on financial sites, and so on.

Each player is interested in maximizing the exposure of his posted item, and may choose his posting time accordingly. Two factors that should be considered are the overall exposure (or viewer potential) of the site, which is generally time-dependent (e.g., depend on the time of day or day in the week), and the posting rate of other players, which compete for the viewer's attention. In our model, each new post lowers the position of previous posts, leading to lower exposure. This interaction between the players leads to a noncooperative game of timing, which we analyze in this paper.

We assume in this paper that the number of players is random and Poisson distributed, fits well the scenario of a large and anonymous population of potential participants (the case of a general distribution is discussed in the technical report [1]). The game takes place on a finite time interval $[0, T]$, where each participating player may choose the submission time of his content according to some probability distribution on that interval. Our focus is on the *symmetric* Nash equilibrium profile, where all players use the same probability distribution. The main results are as follows. We first establish that any equilibrium profile has a continuous distribution over some sub-interval [0,L], and characterize this distribution in terms of an integral equation, or equivalently in terms of a functional differential equation that evolves backward in time. Using the latter characterization we establish existence of an equilibrium profile, and provide a numeric computation procedure that involves an exhaustive search over the single parameter L. We further provide a sufficient condition for uniqueness of the symmetric equilibrium, in terms of a convexity requirement on the relative effectiveness of the list positions. An explicit expression for the equilibrium profile is obtained for the case where the relative effectiveness parameters are geometrically decreasing (which includes in particular the case of a single-item display). We finally provide expressions for the relative social efficiency of the equilibrium solution relative to the social optimum.

Related literature: The closet work to this paper is [17], which considered a similar game model with a single list position. The symmetric equilibrium was explicitly computed for the two-player game, and a discretization scheme was proposed for computing an ϵ-optimal equilibrium for the n-player case. Our model differs from [16] in several respects: (i) A time-varying exposure function $u(t)$ is considered, rather than a constant one. (ii) We allow for multiple list positions. (iii) We consider a Poisson-distributed (rather than deterministic) number of participating players.

Our game belongs to the general class of timing games, where the sole action of each player is a stopping time. Some classical two-player games of timing are reviewed in [6, Chapter 4.5]; more recent pointers may be found in [15]. In the transportation literature, Vickreys bottleneck model [20] addresses strategic timing decisions of commuters who balance road congestion delay with late or early arrival to their destination, where congestion is modeled as a fluid queue. This model has been extensively studied and extended in various directions see, e.g., [16], [11] and their references. A similar model which incorporates a stochastic queue with exponentially distributed service times was introduced in [7], and further extended recently in [9], [12], [10]. It is interesting to note that in these papers, since the customers are lined up in a First Come First Served queue, only players that arrive before us affect our payoff. In the present model the opposite is true only *later* posts affect our payoff.

There are of course many other situations that present a similar timing tradeoff to ours between peak demand periods and the level of peer competition. These may range from finding the best time to share a link on a social network, to marketing oriented decisions of choosing the time to release a new product or launch a publicity campaign. See, for example, the discussions in [19], [18], [5] regarding the seasonality of demand and new product introduction in the food market and in the motion picture industry, and the game formulation of these timing problems in [14], [13]. In the online advertising context, practical guides address the reduction of ad bid prices by avoiding periods of high advertiser competition [2]. Technology updates also involve important questions of timing; in particular, the recent paper [4] studies the timing of cellular network operators' upgrade from 3G to 4G networks.

The paper is organized as follows. Section II presents the system and game model. Section III presents our main results for the general model, including existence, uniqueness, structure, and computation of the equilibrium profile. Section IV treats the special case of geometrically decreasing relative utility parameters, Section V presents some illustrative numerical examples, and Section VI concluded the paper. due to space limitation we only provide here outline of the proofs, and refer the reader to the report [1] for the details.

II. Model Description

We consider an on-line display site, organized as an ordered list, to which users (or players) can post their items during a given time interval $[0, T]$. The list consists of $K \geq 1$ positions of decreasing effectiveness, where position 1 is the most effective. The relative position of posted items is dynamically determined according to their order of arrival: each newly arrived item is placed at the top position, while existing items are pushed one position lower (from 1 to 2, 2 to 3 etc.), and the item at position K (if any) is dropped from the list.

For $t \in [0, T]$ and $k = 1, \ldots, K$, let $u_k(t)$ denote the expected utility rate (per unit time) for a displayed item at time t in position k. The total expected utility over the entire life cycle of an item is therefore

$$U(\mathcal{T}_1, \ldots, \mathcal{T}_K) = \sum_{k=1}^{K} \int_{t \in \mathcal{T}_k} u_k(t) dt\,,$$

where $\mathcal{T}_k$ is the time interval on which the item was displayed at position k. We assume that the functions $u_k(t)$ are decomposed as

$$u_k(t) = r_k u(t), \quad t \in [0, T],\ k = 1, \ldots, K,$$

where

• The *exposure function* $u(t)$, which is common to all positions, captures the the temporal dependence of the utility, due to variation in the exposure of the entire list.

• $(r_1, r_2, \ldots, r_K)$ are the *relative utility parameters*, which capture the relative effectiveness of the different positions on the list. That is, the relative utilities are positive, and are decreasing in the list position k. for convenience, we define $r_{K+1} = 0$.

The following Assumption is imposed throughout the paper.

Assumption 1:

(i) The exposure function $u : [0, T] \to \mathbb{R}$ is continuous and strictly positive, namely $u(t) > 0$ for $t \in [0, T]$. Let $u_{\min} > 0$ and $u_{\max}$ denote the extremal values of u.

(ii) The relative utility parameters (r_k) are decreasing in thelist position. Specifically,

$$r_1 > r_2 \geq r_3 \cdots \geq r_K > 0.$$

The game involves several players, who compete for a place on the list and wish to to maximize their individual utilities. Each player i chooses the posting time t_i of his own item, which is initially placed in the first list position as described above. The number of players in the game is random, and specified as follows:

• The number of players who participate in a given instance of the game is a random variable, denoted $\mathbf{N}_0$. It is assumed here that $\mathbf{N}_0 \sim \text{Pois}(\Lambda)$, a Poisson random variable with parameter $\Lambda > 0$, so that $p(\mathbf{N}_0 = n) = \Lambda^n e^{-\Lambda}/n!$ for $n \geq 0$. We refer to $\mathbf{N}_0$ as the *objective demand*.

• Consequently, under symmetry assumptions, the belief of each participating player regarding the number of *other* players in the game is another random variable, denoted $\mathbf{N}$, which has the same distribution as $\mathbf{N}_0$. We refer to $\mathbf{N}$ as the *subjective demand*. (We note that for a generally-distributed random variable $\mathbf{N}_0$, $\mathbf{N}$ would assume a corresponding length-biased distribution, see [3] or [1] for details. For the Poisson distribution, these two distributions coincide.)

The submission time t_i of player i can be chosen randomly, according to a probability distribution on $[0, T]$ with distribution function $F_i(t), t \in [0, T]$, which represents the (mixed) strategy of this player. Our interest is in the symmetric Nash equilibrium point (NEP) of this game, where all players follow an identical strategy $F_i = F$. The restriction to symmetric strategies, besides its analytical tractability, is natural in the present model where the players are homogeneous and essentially anonymous.

Remark 1: Depending on the circumstances, a player may or may not be able to observe the posting times of other players before posting his own item. Such observations would not affect our results, since under the Poisson assumption above, the distribution of future posts is independent of past posting times.

The Expected Utility: Consider a certain player i who posts his item at time t. Suppose that each of the other $\mathbf{N}$ players uses an identical strategy F. We proceed to calculate the expected utility $U(t; F)$ of the player in that case.

Suppose first that F has no point mass at t, so that with probability 1 there are no simultaneous arrivals at t. Let $N_{(t,s]}$ denote the number of arrivals (by other players) during the time interval $(t, s]$, for $t < s \leq T$. Since i arrives at t, his position on the list at time s will be $k+1$ if $N_{(t,s]}=k$. Consequently,

$$U(t;F) = \mathbb{E}_F\Big(\int_t^T \sum_{k=0}^{K-1} r_{k+1}\mathbf{1}_{\{N_{(t,s]}=k\}}u(s)ds\Big)$$
$$= \int_t^T \sum_{k=0}^{K-1} r_{k+1}\mathbb{P}_F(N_{(t,s]}=k)u(s)ds\,. \quad (1)$$

To compute the probability $\mathbb{P}_F(N_{(t,s]}=k)$, recall that the number of participating players other than i is the random variable $\mathbf{N} \sim \text{Pois}(\Lambda)$. The probability that each of these players posts during the interval $(s,t]$ is $F(s)-F(t)$. Since a Bernoulli dilution of a Poisson RV remains Poisson, it follows that $N_{(t,s]}$ is a Poisson RV with parameter $\Lambda[F(s)-F(t)]$, and

$$\mathbb{P}_F(N_{(t,s]}=k) = \text{Pois}(k;\Lambda[F(s)-F(t)]),$$

where $\text{Pois}(k;\lambda) = \lambda^k e^{-\lambda}/k!$. Substituting in Equation (1) ,we obtain

$$U(t;F) = \int_t^T \sum_{k=0}^{K-1} r_{k+1}\text{Pois}(k;\Lambda[F(s)-F(t)])u(s)ds\,. \quad (2)$$

Simultaneous arrivals: If F has a point mass at t, then there is a positive probability of simultaneous arrivals of several players at that time. In that case we assume that their order of arrival (and subsequent positioning on the list) is determined uniformly at random, and the utility $U(t;F)$ needs to be modified accordingly. However, we need not bother here with the details of this case as we show below that an equilibrium profile F cannot have point masses.

Nash Equilibrium: As usual, a mixed strategy F corresponds to a symmetric Nash equilibrium point (NEP) if F is a best response for each player when all others use the same strategy F. We shall refer to such a strategy F as an *equilibrium profile*.

An equivalent definition of an equilibrium profile, which is more convenient for analysis, requires that $U(t;F)$ be maximized on a set of times t of F-probability 1. That is: there exists a constant u^* and a set $A \subset [0,T]$, such that $\int_A dF(t) = 1$, and

$$U(t;F) = u^* \text{ for } t \in A; \quad U(t;F) \le u^* \text{ for } t \notin A. \quad (3)$$

The equivalence of the two definitions is readily verified. We refer to the value $u^* = u_F^*$ as the *equilibrium utility*.

III. Equilibrium Analysis

We present in this section the main properties of the equilibrium profile that apply to our model.

A. Characterziation

For an arrival profile F and $t \in [0,T]$, denote

$$g(t,F) = \sum_{k=0}^{K-1}(r_{k+1}-r_{k+2})\int_t^T \text{Pois}(k;\Lambda[F(s)-F(t)])u(s)ds\,. \quad (4)$$

Let $F'(t)$ denote the time derivative of F at t, and let *supp*(F) denote the support of the probability measure η_F by a distribution function F.

Theorem 1 (Existence and Characterization): An equilibrium profile F exists, and must satisfy the following properties:

(i) F is a continuous function, and there exists a number $L \in (0,T)$ such that *supp*$(F) = [0,L]$.

(ii) Consequently, a continuous probability distribution function F on $[0,T]$ is an equilibrium profile if, and only if, there exists a number $L \in (0,T)$ such that $F(0)=0$, $F(L)=1$, and $U(t;F)=u_L$ for $t \in [0,L]$ and some constant $u_L > 0$.

(iii) Equivalently, a continuous probability distribution function F on $[0,T]$ is an equilibrium profile if, and only if, there exists a number $L \in (0,T)$ such that: $F(0)=0$, $F(L)=1$, and the derivative $F'(t)$ exists for $t \in (0,L)$ and satisfies the equality

$$F'(t) = \frac{r_1 u(t)}{g(t,F)}\,, \quad t \in (0,L)\,, \quad (5)$$

where $g(t,F)$ is defined in (4).

(iv) For an equilibrium profile F with support $[0,L]$, the equilibrium utility u_F^* is given by $u_F^* = r_1\int_L^T u(s)ds$.

Proof: Due to space limitations we present here a brief outline of the proof. The details may be found in [1].

(i) To argue that F is continuous, suppose that F has an upward jump (i.e., a point mass) at t. This means that there is a positive chance that several ads are submitted simultaneously at t. But then it would be better to submit the ad just after t, leading to $U(t;F) < U(t^+;F)$, which contradicts the assumption that t is in *supp*(F).
To show that *supp*(F) is an interval $[0,L]$, suppose by contradiction that there exists a *gap* in that support, namely numbers $0 \le a < b$ such that $F(a) = F(b) < 1$. We may extend b to the right till it hits *supp*(F). But since there are no submissions in (a,b), it follows that $U(a;F) > U(b;F)$, which means that b cannot belong to the support *supp*(F).

(ii) If F is an equilibrium profile then *supp*$(F)=[0,L]$ by (i), and $U(t;F)=c$ on $t \in [0,L]$ follows by definition of the equilibrium and the continuity of $U(t;F)$ in t. The converse follows by noting that if $F(L)=1$, then there are no arrivals on $(L,T]$ which implies that $U(t;F) < U(L;F) = u_L$ for $t > L$.

(iii) Differentiating the expression (2) for $U(t;F)$ gives $\frac{d}{dt}U(t;F) = -r_1u(t) + F'(t)g(t,F)$. Now, if F is an equilibrium profile then $U(t;F)$ is constant on $[0,L]$, then $\frac{d}{dt}U(t;F)=0$ on $(0,L)$, and we obtain the differential relation (5). The converse statement follows by arguing that (5) implies that $U(t;F)$ is constant on $[0,L]$.

Part (iv) of the Theorem is straightforward, upon noting that there are no new submissions on $(L,T]$ so that an ad submitted at $t=L$ remains at the top position till T.

The proof of *existence* of an equilibrium relies on the differential characterization in part (iii). For that purpose, we consider Equation (5) as a (functional) differential equation with terminal conditions $F(t)=1$ for $t \in [L,T]$. It is argued that the solutions $F_L(t)$ to this equation are well defined and continuous in L (Proposition 4). From this we deduce that

$F_L(0) = 0$ for some L, which implies that the corresponding solution $F_L(t)$ is an equilibrium profile. ■

Remark 2: The differential relation in (5) provides a *functional* differential equation for F, as the right-hand side depends on entire function F and not only on its value at t. This relation provides the basis for the existence and uniqueness claims in this and the next subsection, as well as for the computational procedure that follows.

Remark 3: The results of the last Theorem do not imply uniqueness of the equilibrium profile. However, if multiple equilibria do exist, we obtain a strict ordering among them. First, we note that to each number L and support $[0, L]$, there correspond at most one equilibrium profile (this follows by uniqueness of solutions to Equation (5), see Proposition 4 below). Consider now two equilibrium profiles F_1 and F_2, indexed by L_1 and L_2 respectively, with $L_1 < L_2$. By part (iv) of the Theorem it follows that the equilibrium utility u^* of F_1 is strictly higher than that of F_2. That is: the equilibrium with the smaller support $[0, L]$ is better (in terms of individual utilities).

B. Uniqueness

To establish uniqueness of the equilibrium profile, we require an additional condition on the relative utilities (r_k). We observe that this is only a sufficient condition, and uniqueness may well hold in greater generality.

Assumption 2: The relative utility parameters satisfy the following convexity condition:

$$r_k \leq \frac{1}{2}(r_{k-1} + r_{k+1}), \quad k = 2, \ldots, K \tag{6}$$

(recall that $r_{K+1} = 0$ by definition).

Theorem 2 (Uniqueness): Suppose Assumption 2 holds, then the equilibrium profile F is unique.

We note that Assumption 2 holds trivially for a single list position, namely $K = 1$. It also holds for linearly-decreasing utilities of the form $r_1 = (K + 1)d, \ldots, r_K = d$ for some $d > 0$, as well as for the case of geometrically decreasing utilities that we consider in the next section.

The proof of Theorem 2 relies on the following monotonicity properties of the function g defined in (4).

Lemma 3: Let Assumption 2 hold.

(i) Let F_1 and F_2 be two strategy profiles (namely distribution functions over $[0, T]$) such that, for some $t \in [0, T)$,

$$F_2(s) - F_2(t) \geq F_1(s) - F_1(t) \quad \text{for all } s \in [t, T]. \tag{7}$$

Then $g(t, F_2) \leq g(t, F_1)$.

(ii) If, in addition, the inequality in (7) is strict over some nonempty interval $I \subset [t, T]$, then either $g(t, F_2) < g(t, F_1)$, or else $g(\tau, F_2) = g(\tau, F_1)$ for all $\tau \in [0, T]$.

Proof: Denote $\delta_k = r_k - r_{k+1}$. Observe from (4) that

$$g(t, F) = \int_t^T \left(\sum_{k=0}^{K-1} \delta_{k+1} \mathrm{Pois}(k; \Lambda[F(s) - F(t)]) \right) u(s) ds.$$

We will show that the integrand is decreasing in the difference $F(s) - F(t)$. Since $u(s) > 0$, we need consider only the expression in the larger brackets.

Fix s and t, and substitute λ for $\Lambda(F(s) - F(t))$ in this expression. That is, consider

$$f(\lambda) = \sum_{k=0}^{K-1} \delta_{k+1} \mathrm{Pois}(k; \lambda) = \sum_{k=0}^{K-1} \delta_{k+1} \frac{\lambda^k}{k!} e^{-\lambda}.$$

Differentiating with respect to λ and rearranging gives

$$\begin{aligned} f'(\lambda) &= \sum_{k=0}^{K-1} \delta_{k+1} \left(\frac{\lambda^{k-1}}{(k-1)!} - \frac{\lambda^k}{k!}\right) e^{-\lambda} \\ &= \sum_{k=0}^{K-1} (\delta_{k+2} - \delta_{k+1}) \frac{\lambda^k}{k!} e^{-\lambda}. \end{aligned}$$

But Assumption 2 implies that $\delta_{k+1} - \delta_k = 2r_{k+1} - r_k - r_{k+2} \leq 0$, so that $f'(\lambda) \leq 0$. Part (i) of the Lemma clearly follows. Part (ii) follows by observing from the last expression that either $f'(\lambda) = 0$ for all λ, or $f'(\lambda) < 0$ for all $\lambda > 0$. ■

Proof of Theorem 2: Recall that an equilibrium profile F satisfies the properties in Theorem 1 with some parameter L. In particular, $F(t) = 1$ on $t \in [L, T]$, F satisfies equation (5) on $[0, L]$, and $F(0) = 0$. Let F_1 and F_2 denote two equilibrium profiles with corresponding parameters L_1 and L_2. We will show below that $L_1 < L_2$ implies that $F_2(0) < F_1(0)$, so that only one can be an equilibrium. Therefore $L_1 = L_2$. But Proposition 4 implies that L defines F uniquely, hence the equilibrium is unique.

Consider then F_1 and F_2 as above, and suppose that $L_1 < L_2$. Since $F_1(s) = 1$ for $s \in [L_1, T]$ and F_2 is strictly increasing over $t < L_2$, it follows that

$$F_2(L_1) < F_1(L_1) = 1, \tag{8}$$

and

$$F_2(s) - F_2(L_1) > F_1(s) - F_1(L_1) = 0, \quad s \in (L_1, T]. \tag{9}$$

Therefore inequality (7) is satisfied with strict inequality for $t = L_1$. By Lemma 3(ii), exactly one of the following two conclusions holds:

(a) $g(t, F_2) = g(t, F_1)$ for all $t \in [0, T]$. In that case it follows from (5) that $F_2'(t) = F_1'(t)$ holds for $t \leq L_1$, so that

$$F_1(0) - F_2(0) = F_1(L_1) - F_2(L_1) > 0.$$

(b) $g(t, F_2) < g(t, F_1)$ at $t = L_1$. By (5), this implies that $F_2'(L_1) > F_1'(L_1^-)$. We argue that this inequality extends to all $t < L_1$. Suppose, to the contrary, that $F_2'(\tau) \leq F_1'(\tau)$ for some $\tau < L_1$. Noting that F_1' and F_2' are continuous on $[0, L_1)$ by (5), there must exist a time $t_0 < L_1$ so that $F_2'(t_0) = F_1'(t_0)$, while $F_2'(s) > F_1'(s)$ for $s \in (t_0, L_1]$. By integration, it follows that

$$F_2(s) - F_2(t_0) \geq F_1(s) - F_1(t_0), \quad s \in [t_0, L_1].$$

Combined with (9), we may apply Lemma 3(ii) to deduce that $g(t_0, F_2) < g(t_0, F_1)$, hence $F_2'(t_0) > F_1'(t_0)$. But this contradicts the definition of t_0. We have thus verified that $F_2'(t) > F_1'(t)$ for all $t < L$. But this implies that

$F_2(0) - F_1(0) < F_2(L_1) - F_1(L_1) < 0$, where the first inequality follows by integration, and the second from (8).

We have thus shown that $L_1 < L_2$ implies $F_2(0) < F_1(0)$, which completes the proof of Theorem 2. ■

C. Computation

We outline next a numeric procedure for computing all equilibrium profiles. This procedure relies on computing the solution F of the functional differential equation (FDE) (5) for different values of the parameter $L \in (0, T)$, and searching for values of L for which the boundary conditions $F(L) = 1$ and $F(0) = 0$ are satisfied. The latter essentially involves an exhaustive search over the scalar parameter L. When the uniqueness condition in Assumption 2 is satisfied, that search can be expedited by observing monotonicity properties of the solution F in L.

For each $L \in (0, T)$, we consider the differential equation (5), with terminal conditions $F(t) = 1$, $t \in [L, T]$. As mentioned, this is a *functional* differential equation since the derivative $F'(t)$ at time$t$ depends on values of F at other times as well. However, a key property of that equation, which follows from the definition of $g(t, F)$ in (4), is that this dependence is one-sided: $F'(t)$ depends only on 'future' values of F, namely on $(F(s), s \geq t)$. In the terminology of [8], this equation is a *retarded* FDE (up to time reversal). This property allows to back-integrate this equation, starting with the above-mentioned terminal conditions, and proceeding backward in time.

We collect some properties of the FDE (5) in the following Proposition. These support the numerical computation procedure that follows, and are also used in the proofs of the pervious results.

Proposition 4: For $L \in (0, T)$, define $F_L(t) = 1$ for $t \in [L, T]$. Consider the FDE (5) for F_L, namely

$$F'_L(t) = \frac{r_1 u(t)}{g(t, F_L)}, \quad t \leq L \tag{10}$$

where g is defined in (4).

(i) The FDE (10) admits a unique solution $F_L(t)$ over $t \in [t_L, L]$, where

$$t_L = \inf\{s \in [0, L) : F_L(s) > 0\}.$$

(ii) t_L is a continuous function of L, and $F_L(t)$ is a continuous function of L for each $t \in (t_L, L]$.

Note that t_L is the first time $t \geq 0$ at which $F_L(t)$ becomes zero. As $g(t, F_L)$ is meaningless for $F_L < 0$, the solution of equation (10) cannot be extended beyond this point. For the sake of exposition, it will be convenient to linearly extend $F_L(t)$ below t_L when $t_L > 0$, using

$$F_L(t) = -(t_L - t), \quad t \in [0, t_L]\,.$$

This gives $F_L(0) < 0$ when $t_L > 0$.

The equilibrium profiles now correspond to those values of L for which $F_L(0) = 0$. To search for these values, we may use the following crude exhaustive search approach:

- For values of L in a grid over $(0, T)$, integrate equation (10) numerically (using Euler approximation or a more accurate method) and obtain $F_L(t)$, $t \in [0, T]$.
- The equilibrium points correspond to values of L where $F_L(0) \simeq 0$.

The search grid may of course be refines around points of interest. We note that under the uniqueness condition (6), we know that $F_L(0)$ crosses 0 at a unique value of L, which clearly simplifies the search.

Remark 4: We observe that if $F(t)$ is an equilibrium profile for the model with the given exposure function $u(t)$, then $\tilde{F}(t) = F(g(t))$ is an equilibrium profile for the model with unit exposure function, $u(t) \equiv 1$, where $g(t)$ denotes the time change

$$g(t) = \frac{T}{\int_0^T u(s)ds} \int_0^t u(s)ds\,.$$

This follows by verifying, through a change of integration variable, that $U(t; F)$ in equation (2) is equal to $U(g(t), \tilde{F})$ in the model with $u = 1$. Therefore, once the equilibrium profile is computed for one exposure function $u(t)$, it can be readily obtained for any other.

D. Social Efficiency

Under an equilibrium profile F with support *supp*$(F) = [0, L]$, the expected utility u_F^* per player is given by Theorem 1(iv). Recalling that $\mathbf{N}_0 \sim \text{Pois}(\Lambda)$ is the objective demand, the expected social utility may be seen to be

$$S(F) = E(\mathbf{N}_0)u_F^* = \Lambda r_1 \int_L^T u(s)ds\,.$$

Let us compare that to the optimal social utility. Under full utilization, namely all K positions occupied over $[0, T]$, the social utility is clearly

$$\sum_{k=1}^{K} r_k \int_{s=0}^{T} u(s)ds\,.$$

However, taking into account that if $n < k$ player arrive then some lower positions remain empty, the optimal social utility becomes

$$S^* = \sum_{k=1}^{K} r_k p(\mathbf{N}_0 \geq k) \int_{s=0}^{T} u(s)ds\,. \tag{11}$$

The *relative efficiency* of the equilibrium is now defined and evaluated by the ratio:

$$\rho(F) \triangleq \frac{S(F)}{S^*} = \frac{r_1 \Lambda}{\sum_{k=1}^{K} r_k p(\mathbf{N}_0 \geq k)} \frac{\int_L^T u(s)ds}{\int_0^T u(s)ds}\,. \tag{12}$$

This expression still depends on the equilibrium parameter L. Explicit expressions for some specific cases will be derived in the following sections.

IV. Geometric Utilities

We now turn to consider the specific case where the list is infinite, and the relative utility parameters (r_k) are geometrically decreasing: $r_k = r_1 a^{k-1}$ for some $0 \leq a < 1$. That is, that relative utility of the list positions in decreasing at a fixed ratio. In this case the obtained equilibrium profile is explicit and has a clear intuitive appeal.

In the special case of $a = 0$ we obtain that $r_k = 0$ for $k \geq 2$, which coincides with the case of a single list position ($K = 1$). We start the exposition by considering this simple case first, and then show how the solution for the general case ($a \geq 0$) can be reduced to this case.

A. Single List Position

Consider first the case of a single list position, $K = 1$, with utility function $u_1(t) = r_1 u(t)$. From (2) we obtain

$$U(t;F) = r_1 \int_t^T e^{-\Lambda(F(s)-F(t))} u(s) ds \tag{13}$$

We proceed to derive (5) directly. By differentiating the last expression,

$$\frac{d}{dt}U(t;F) = -r_1 u(t) + r_1 \Lambda F'(t) \int_t^T u(s) e^{-\Lambda[F(s)-F(t)]} ds \,.$$

Since $\frac{d}{dt}U(t;F) = 0$ on $(0, L)$, we obtain

$$F'(t) = \frac{r_1 u(t)}{\Lambda r_1 \int_t^T e^{-\Lambda[F(s)-F(t)]} u(s) ds}, \quad t \in (0, L)\,, \tag{14}$$

which is the explicit form of (5) in this case.

Observe now that the denominator of the last expression is just Λ times $U(t;F)$, while the latter identically equals u_F^* on $[0, L]$ by the basic equilibrium property. Therefore,

$$F'(t) = \frac{r_1}{\Lambda u_F^*} u(t), \quad t \in (0, L)\,. \tag{15}$$

We see that $F'(t)$ is proportional to $u(t)$ on its support!

It remains to determine L and u_F^*. Recall from Theorem 1 that

$$u_F^* = U(L;F) = r_1 \int_L^T u(s) ds\,. \tag{16}$$

On the other hand, by (15) and the definition of L,

$$1 = F(L) = \int_0^L F'(s) ds = \frac{r_1}{\Lambda u_F^*} \int_0^L u(s) ds\,,$$

which implies

$$u_F^* = \frac{r_1}{\Lambda} \int_0^L u(s) ds\,. \tag{17}$$

Comparing the last two expressions for u_F^*, we arrive at the following equation, from which L (and hence u_F^*) can be computed:

$$\int_L^T u(s) ds = \frac{1}{\Lambda} \int_0^L u(s) ds \tag{18}$$

Observe that this equation has a unique solution, since the left-hand side in continuously and strictly decreasing to zero, while the right-hand side is continuously increasing from zero.

We summarize these finding in the following Proposition:

Proposition 5 (Equilibrium Profile): For the case of Poisson demand with mean Λ and a single list position ($K = 1$), the equilibrium profile F is given in terms of its density F' by

$$F'(t) = \frac{1}{\int_0^L u(s) ds} u(t), \quad t \in (0, L)\,, \tag{19}$$

where L is the unique solution in $(0, T)$ of equation (18).

Example 1: For a simple example, consider the case of time-invariant exposure rate, $u(t) \equiv u_0$. Here we obtain that the equilibrium profile corresponds to a uniform distribution on $[0, L]$, and from (18) we obtain $(T - L) = L/\Lambda$, so that $L = \frac{\Lambda}{\Lambda+1} T$. Note that for $\Lambda \to \infty$ (heavy load), $L \to T$, i.e., the support of the equilibrium profile extends to the entire interval $[0, T]$, as may be expected. This in fact holds for any exposure function u, as may be seen from (18).

Remark 5: It may be seen from (19) that the equilibrium density $F'(t)$ is proportional to the exposure function $u(t)$. This appears intuitively appealing – higher arrival probabilities are assigned to times with larger utility. However, it is actually remarkable that the arrival rate at time t depends on u only through $u(t)$ at t, and not at later times. This *myopic* appearance of the equilibrium is a result of the Poisson distribution of the demand.

We turn to consider the efficiency of the Nash equilibrium in this case, which can be computed in closes form.

Proposition 6 (Efficiency): The equilibrium utility is given by

$$u_F^* = \frac{1}{1+\Lambda} r_1 \int_0^T u(s) ds\,.$$

Consequently, the relative efficiency at equilibrium, as defined in (12), equals

$$\rho(F) = \frac{1}{(1+\Lambda^{-1})(1-e^{-\Lambda})}\,.$$

Proof: The expression for the equilibrium utility follows from (16)-(17), which imply that

$$\int_0^T u(s) ds = \frac{\Lambda u_F^*}{r_1} + \frac{u_F^*}{r_1} = \frac{1+\Lambda}{r_1} u_F^*\,.$$

Noting that $E(D_0) = E(D) = \Lambda$ under the Poisson distribution, we obtain $S(F) = \Lambda u_F^*$. Next, evaluating (11) with $K = 1$ and $\sum_{n \geq 1} = 1 - p_0(n) = 1 - e^{-\Lambda}$, we obtain

$$S^* = (1 - e^{-\Lambda}) r_1 \int_0^T u(s) ds\,.$$

Substituting $S(F)$ and S^* in (12) gives the required ratio $\rho(F)$. ■

Some observations are in order regarding last result.

- It is remarkable that the relative efficiency $\rho(F)$ does not depend on the time-dependent exposure function $u(t)$, but only on the mean demand Λ.
- The relative efficiency $\rho(F)$, which is smaller than 1 by definition, converges to 1 both when $\Lambda \to 0$ (where essentially at most one player shares the system) and, more importantly, when $\Lambda \to \infty$ (heavy demand).

B. Infinite List with Geometric Utilities

The results of the previous Subsection can be readily be extended to the asymptotic model of an infinite List, $K = \infty$, with geometrically decreasing relative utility parameters: $r_k = r_1 a^{k-1}$, $k \geq 1$, with $0 < a < 1$.

Proceeding formally, by substituting r_k above in (2) and evaluation the resulting geometric series, we obtain

$$U(t; F) = r_1 \int_t^T e^{-\Lambda(1-a)(F(s)-F(t))} u(s) ds \,. \tag{20}$$

It may be seen that (20) is identical to (13), once $\Lambda(1-a)$ is substituted for Λ. Thus, all the results of the previous subsection regarding the equilibrium profile F and equilibrium utility u_F^* are valid in the present model as well, after this substitution.

The relative efficiency can also be evaluated using a similar substitution. To see this, observe that

$$S(F) = \Lambda u_F^* = \frac{\Lambda}{1 + \Lambda(1-a)} r_1 \int_0^T u(s) ds \,.$$

Also, evaluating (11) can be seen to give in this case

$$S^* = \frac{1 - e^{-\Lambda(1-a)}}{1-a} r_1 \int_0^T u(s) ds \,.$$

Therefore,

$$\rho(F) = \frac{S(F)}{S^*} = \frac{1}{(1 + x^{-1})(1 - e^{-x})} \,, \quad x \triangleq \Lambda(1-a) \,.$$

V. A Numerical Example

We next present some numerical computations that illustrate the equilibrium profile obtained for different choices of model parameters, and compare these results to our theoretical findings. The results are obtained using the computational procedures outlined in Section V, implemented in MATLAB. The basic Euler method is used to integrate the relevant differential equations.

We assume here a uniform exposure function, $u(t) = 1$ for $t \in [0, T]$. In view of Remark 4, the results for other exposure functions can be obtained by an appropriate time change. We further set $T = 10$ and $\Lambda = 3$.

Several values for the list size K were considered, with relative utility parameters decreasing geometrically according to $r_k = (0.5)^{k-1}$, $1 \leq k \leq K$. Thus for $K = 1$ we have $r_1 = 1$, and for $K = 3$ we have $(r_1, r_2, r_3) = (1, \frac{1}{2}, \frac{1}{4})$.

The results obtained for $K = 1, 2, 3, 6$ are depicted in Figure 1. For $K = 1$, the results confirm the theoretical prediction in Proposition 5, namely a uniformly-distributed equilibrium profile with L solving equation (18). Since $u = 1$ this equation gives $(10 - L) = L/3$, or $L = 7.5$.

For $K = 6$, since the r_k's decay exponentially, the model can be approximated by the asymptotic model with $K = \infty$ that was studied in Subsection IV-B. As observed there, the obtained equilibrium is identical to the one obtained for a single-position case, with an effective demand of $\Lambda' = \Lambda(1 - a) = 1.5$. Thus, we expect a uniform distribution on $[0, L]$,

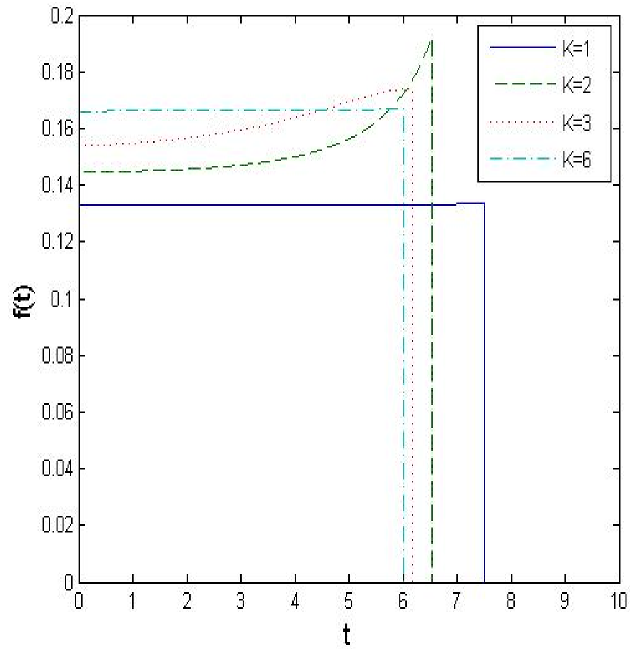

Fig. 1: Equilibrium density functions.

with L determined from $(10 - L) = L/1.5$, which gives $L = 6$. This is indeed what was obtained numerically.

In contrast to the above extreme case, for for intermediate values of $K = 2$ and $K = 3$, the equilibrium distribution may be seen to be non-uniform.

VI. Conclusion

We have considered in this a stylized model of advertiser timing competition, where advertisers (or other media users) compete for position over a shared publication medium that gives priority to recently submitted items. Our results address the existence, uniqueness, and computation of the symmetric equilibrium profile, as well as some explicit solutions for certain special cases of the model.

The basic model of this paper may be enriched in several directions, that include:

- Additional cost components, such as a one-time submission cost.
- Long-rage periodic model: Our model assumes a finite and per-specified display interval $[0, T]$. It is of practical interest to consider a variant of this model that operates on a long (indefinite) duration, with a continuous supply of new ads (which may fluctuate periodically, say on a daily basis, along with the exposure function $u(t)$).
- Large-demand limits: When the number of submitted ads is large, fluid-scale models that assumes a deterministic rather than stochastic submission rates may be computationally attractive. The analysis of such fluid models and their relations to the stochastic ones are of interest here.

- Multi-class models: It may be of interest to consider several classes of advertisers with different preferences – for example, the exposure function $u(t)$ may differ among advertisers that address different segments of the population.

These topics present challenges with various degrees of difficulty, which are left for future investigation.

VII. ACKNOWLEDGMENTS

This research was supported by the Israel Science Foundation (ISF) grant no. 1319-11. We wish to thank the referees for their useful comments and for pointing out several typos in the original manuscript.

REFERENCES

[1] A. Altman and N. Shimkin, Strategic Timing of Item Posting over a Shared Medium, Technical Report, available from http://webee.technion.ac.il/people/shimkin/PAPERS/... StrategicTimingRep2014.pdf

[2] J. Constine, "Minimizing Your Facebook Ad CPC Bid Prices by Avoiding Bid Competition," http://www.insidefacebook.com/2011/09/08/suggested-bid-cpc/, Sept. 2011.

[3] R. B. Cooper, *Introduction to Queueing Theory,* 2nd Ed., North-Holland, New York, 1981.

[4] L. Duan, J. Huang and J. Walrand, "Economic analysis of 4G network upgrade," Proc. INFOCOM 2013, pp. 1070-1078.

[5] L. Einav, "Seasonality in the U.S. Motion Picture Industry," *RAND Journal of Economics* 38(1):127-45, 2007.

[6] D. Fudenberg and J. Tirole, *Game Theory,* MIT press, Cambridge, MA, 1991.

[7] A. Glazer and R. Hassin, "?/M/1: On the equilibrum distribution of customer arrivals," *European J. of Oper. Research* 13:146-150, 1983.

[8] J. K. Hale and S. M. Verduyn Lune, *Introduction to Functional Differential Equations,* Springer, New York, 1993.

[9] R. Hassin and Y. Kleiner, "Equilibrium and optimal arrival patterns to a server with opening and closing times," *IEE Transactions* 43(3):164-175, March 2011.

[10] M. Haviv, "When to arrive at a queue with tardiness costs?" *Performance evaluation* 70:387-399, 2013.

[11] R. Jain, S. Juneja and N. Shimkin, "The Concert Queueing Game: To Wait or to be Late," *Discrete Event Dynamic Systems* 21(1):103-138, April 2011.

[12] S. Juneja and N. Shimkin, "The Concert Queueing Game: Strategic arrivals with waiting and tardiness costs," *Queueing Systems* 74(4):369-402, August 2013.

[13] T. Klastorin and W. Tsai, "New Product Introduction: Timing, Design, and Pricing," *Manufacturing and Service Operations Management* 6(4):302-320, 2004.

[14] R. E. Krider and C. B. Weinberg, "Competitive Dynamics and the Introduction of New Products: The Motion Picture Timing Game," *Journal of Marketing* 35(1):1-15, 1998.

[15] R. Laraki, E. Solan and N. Vieille, "Continuous-time Games of Timing," *Journal of Economic Theory* 120(2):206-238, February 2005.

[16] R. Lindley, "Existence, uniqueness, and trip cost function properties of user equilibrium in the bottleneck model with multiple user classes," *Transportation Science* 38(3):293-314, 2004.

[17] Z. Lotker, B. Patt-Shamir, and M. R. Tuttle, "A game of timing and visibility," *Games and Economic Behavior* 62(2):643-660, 2008.

[18] S. Radas and S. M. Shugan, "Seasonal Marketing and Timing New Product Introductions," *Journal of Marketing Research* 35(3):296-315, 1998.

[19] D. Rajagopal, "New Product Introduction and Seasonality Effect in Food Products Retailing," *Journal of Food Products Marketing* 14(4):54-75, 2008.

[20] W. S. Vickrey, "Congestion theory and transport investment," *The American Economic Review* 59:251-260, 1969.